The Generalized Euler Equation and the Bankruptcy-Sovereign Default Problem*

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Abstract

We characterize the equilibrium of the standard sovereign default model where a risk-averse borrower issues long-term non-contigent bonds but cannot commit its future selves to repay. We show existence and uniqueness of the Markov equilibrium of the dynamic game with successive borrowers that is associated to this environment. We show that the price and policy functions exhibit jumps and kinks in various places. A suitable choice of arbitrary small noise yields price and policy functions that are differentiable almost everywhere which allows us to characterize the equilibrium using only decision rules of the agents by means of a set of functional equations. Further, we describe the equilibrium objects via an Euler equation with derivatives on future actions —i.e. a generalized Euler equation (GEE) where the effects due to default and those to dilution can be disentangled. Computational strategies using these functional equations allow for solving the model with continuous functions using policy iterations. A sufficient variance of the noise allows for concavity and hence unique solution of the GEE which eases computation.

Keywords: Long term debt, Sovereign default, Generalized Euler Equation

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1 Introduction

The use of long-term debt (in the spirit of Hatchondo and Martinez (2009), Arellano and Ramanarayanan (2012) and Chatterjee and Eyigungor (2012)) has become omnipresent in the literature on sovereign debt. It is characterized by competitive risk-neutral lenders with deep pockets that trade long-term non-contingent debt contracts with a risk-averse borrower that can always choose to default. The borrower's income is stochastic, so there is value to accessing credit markets. Nevertheless, the borrower cannot credibly commit its successors (or future selves) to repay which affects the price of debt, a feature that requires thinking in terms of equilibrium rather than of a maximization problem.

This is a particularly fiendish problem as we will see below. In its original form, consumption and bond holdings policy functions do have discontinuities, kinks and flat areas. Accordingly, these models are solved numerically by discretization of the choice and state spaces, and even in these circumstances, convergence to a solution is not typically achieved without the use of randomization devices (Chatterjee and Eyigungor (2012) and Dvorkin et al. (2021)).

The purpose of this paper is to provide a precise characterization of the equilibrium, its existence, uniqueness and, in particular, of the elements that determine the trade-offs faced by the borrower. In addition, we show how to ease the computation of these models while increasing precision and doing away with the issues induced by discretization.

We first lay down the original decision problem and show that the bond price admits points of non-differentiability which affect the optimal borrowing decision. We proceed by posing extreme value shocks (or in general any continuous shocks) to the default decision, hereby transforming the decision problem to allow for randomization between repayment and default.

In this environment, we establish that there are two types of equilibrium behavior in the decision problem. In the first, the borrower chooses to never exceed the risky debt limit, and so does not face a discount for dilution risk. In the second, the borrower enters the risky borrowing region with positive probability, so all debt levels have a discount for dilution risk. The presence of those two behaviors raises the question of whether these are unique equilibria for economies with different primitives, or whether there is multiplicity in the long-term debt model.

Taking the limit of finite horizon economies, we show that an equilibrium exists and is unique with the same properties than those of finite horizon economies. This unique equilibrium is a Markov equilibrium. In other words, we rule out the type of trigger strategy equilibria coded as a Markov-like equilibria described by Krusell and Smith (2003). In doing so, our approach mimics the one of Bhaskar et al. (2012) in repeated games.

We characterize the optimality conditions analytically by means of an intertemporal first-order condition with derivatives on future actions —i.e. a Generalized Euler Equation (GEE). We proceed by first assuming convexity of the decision problem and differentiability of the price, the default set and the bond policy function to derive the GEE, and then we provide conditions under which these two assumptions hold true. The GEE enables a complete characterization of interior solutions and determines the main forces at play. We show that the introduction of extreme value shocks can convexify the decision problem and enables the price and the policy functions to become smooth everywhere. Hence, the proper derivation of the GEE is possible for all positive values of debt. Even in the case that the problem is not convex, the GEE is still very useful, albeit it has more than one solution, and computation requires that we also keep track of the value function to make sure that we pick the global maxima.

The starting point of the literature on sovereign debt and default is the study of Eaton and Gersovitz (1981) which has been directly extended by Aguiar and Gopinath (2006) and Arellano (2008).[1] The initial model considers a sovereign borrower which trades one-period bonds with a continuum of competitive lenders. In this environment, Hatchondo and Martinez (2009), Arellano and Ramanarayanan (2012) and Chatterjee and Eyigungor (2012) introduce long-term debt in the form of a geometrically decreasing maturity rate. This approach is computationally friendly as it cuts down on the number of state variables and enhances the quantitative fit to the data. Nevertheless, analytical characterization of the problem becomes more arduous as the bond price depends on future actions.

Our analysis provides an analytical characterization of incomplete markets models with default under long-term bonds. Such characterization already exists for short-term debt. More precisely, Auclert and Rognlie (2016) show existence and uniqueness in Eaton and Gersovitz (1981). The proof relies on the standard argument made by Bulow and Rogoff (1989) that reputation alone cannot sustain debt. Feng and Santos (2021) extend the existence and uniqueness results to a model with capital, and furthermore show existence of an equilibrium with smooth policies with the addition of taxation. Aguiar et al. (2019) also show existence and uniqueness by looking at the dual problem of the revenue maximizing lender. Finally, Clausen and Strub (2020) use reverse calculus and nested optimization. Their paper is the closest in spirit to ours as they also use differentiability to characterize equilibrium, albeit of short term debt (and other environments such as adjustment costs and social insurance). In our environment that long-term bonds introduce the possibility to dilute legacy debt which requires the bond policy to be differentiable alongside the bond price and the default set. With short-term debt, differentiability of the

[1] There is an associated large literature on household bankruptcy that is based on essentially the same theory and is supported by the U.S. Bankruptcy code that justifies many of the assumptions made (see for instance Livshits et al. (2007) or Chatterjee et al. (2007)).

bond policy is not needed. Another important difference between is that in Clausen and Strub (2020) the lack of smoothing shocks prevent the use of the GEE because of kinks that have to be found. Their results only speak of situations where the first order condition holds as an equality.

Regarding long-term debt, Chatterjee and Eyigungor (2012) show the existence of an equilibrium bond price. Their proof relies on randomization through continuous *i.i.d.* shocks and restricting the choice of debt to a finite set. This allows for the application of Brouwer's Fixed Point Theorem. Subsequently, Aguiar and Amador (2020) show that there exist two equilibria in such class of model: a "borrowing" equilibrium where the borrower issues debt until it reaches some debt limit and defaults, and a "saving" equilibrium where the borrower reduces the stock of debt until default no longer occurs with positive probability. In contrast, we show uniqueness by focusing on equilibria that are the limit of those of ficanite economies, while Aguiar and Amador (2020) might not rule out trigger strategy equilibria coded as a Markov-like equilibria.

Our characterization relies on the GEE. More precisely, the Euler equation contains derivatives of prices. We therefore relate to the work on time inconsistent policy of notably Krusell et al. (2002, 2010), Krusell and Smith (2003), Klein et al. (2008), Mateos-Planas (2010) and others. Our work relates to the work of Hatchondo and Martinez (2009), Niepelt (2014), Arellano and Ramanarayanan (2012), and Hatchondo et al. (2016) who study the trade-off between the issuance of short-term and long-term debt and to the work of Arellano et al. (2023) who analyze the decision to partially default. An early version of Aguiar and Gopinath (2006) and Hatchondo et al. (2016) include also a discussion of the first order conditions using debt price derivatives. All the aforementioned studies adopt a heuristic approach by assuming that the relevant policy functions and prices are differentiable ignoring the existence of the critical points or thresholds discussed above. In opposition, we show that those objects are indeed differentiable everywhere provided the default decision is affected by utility shocks.

We use the GEE as a basis for computations. This allows us to solve for the policy functions using global methods that avoid discretization. Specifically, we use piecewise cubic Hermitian polynomials to approximate policy and value functions as well as prices. The reason is that these functions use derivatives in their construction and hence their implied derivatives outside the grid can be used in the evaluation of the GEE. This allows us to compute the equilibrium accurately and fast, a feature that is not always possible when using discretization of the state space (even if extreme value shocks are used to avoid oscillations). Hence, we also address the literature on quantitative methods to solve models of incomplete markets with default. Hatchondo et al. (2010) show that different solution methods such as splines or Chebyshev polynomials lead to different numerical results than the standard value function iteration algorithm in the case of short-term debt. Closer to our analysis, Arellano et al. (2016) use the Euler equation to solve short-term debt problem numerically, but assume the GEE always holds. Our characterization applies to the case of long-term debt and suggests the use of small extreme value shock perturbations to ensure the validity of the GEE everywhere. Thus, our characterization of the GEE properly relies on two main ingredients: the limit of finite horizon and extreme value shock perturbations. Moreover, the set of solutions to the GEE will not be unique when the problem is not convex (because of insufficient utility noise), yet the relevant solution to the GEE (the one associated to the global maximum) tells us how the different elements weight in the optimal choice.

We rely on extreme value shocks to solve the model with the GEE by means of the endogenous grid method algorithm developed by Carroll (2006).Unlike Arellano et al. (2020), Mihalache (2020) and Dvorkin et al. (2021), we only introduce such shocks in the value of defaulting and repayment. We show that there is a variance of these shocks that is sufficient to ensure strict convexity of the maximization problem. Contrary to the model with employment lotteries of Rogerson (1988), the decision problem might remain non-convex with small scale parameters. Finally, we show that extreme value shocks enable the differentiability of the bond price and the bond policy function.

The paper is organized as follows. Section 2 presents the environment and the decision problem. Section 3 considers the decision problem under extreme value shocks and 4 additionally bounds the horizon. Thereafter, Section 5 derives the GEE and characterizes the equilibrium. Section 6 presents the quantitative analysis including details of the relative performance of various methods. Section 7 concludes. All the proofs are in the Appendix.

2 The Model

We consider a standard model of incomplete markets with default in the spirit of Eaton and Gersovitz (1981). A risk averse sovereign trades long-term bonds with deep pockets risk neutral lenders. Crucially, it cannot commit its successors (or future selves) to repay what it owes. We start with the environment and decisions (Section 2.1) before defining the bond price and the Markov equilibrium (Definition 1).

2.1 Environment and Decisions

There are competitive, risk neutral lenders that discount the future at rate $\overline{p} \equiv \frac{1}{1+r}$ where $r = R - 1$ is the exogenous world risk-free interest rate.

The sovereign receives a stochastic endowment $y \in Y \equiv [\underline{y}, \overline{y}] \subset \mathbb{R}^{++}$ which is $i.i.d.$ distributed with *absolutely continuous* distribution function $F_y(y)$ and density $f_y(y) > 0$.[2] The sovereign has preferences

[2] The $i.i.d.$ assumption is to reduce notation. Given that there is no private information, the extension to a Markovian

represented by a strictly concave, monotone and bijective utility function $u : \mathbb{R}^{++} \to \mathbb{R}$ of class $\mathcal{C}^{\infty}$ and discounts the future at rate $\beta < \overline{p}$.

The sovereign has access to non-contingent bonds $b \in B \equiv [\underline{b}, \overline{b}]$ where $b > 0$ represents a debt and $b < 0$ an asset. Bonds are long-term and follow the structure of Hatchondo and Martinez (2009): each period a coupon of one is paid per unit of debt and a fraction $\lambda \in [0, 1)$ of the debt matures.

In addition, we assume there is some arbitrarily large finite value of debt $\overline{b} > \frac{R\overline{y}}{1-R}$ greater than the present value of the maximum endowment, and similarly consumption $\overline{c} > \frac{R\overline{y}}{1-R}$ that bounds the sovereign's choice set. Together these assumptions imply the value function is bounded by $\overline{V} = \frac{u(\overline{c})}{1-\beta}$. This also implies there is an upper bound $-\underline{b} = \frac{R}{R-1}(\overline{c} - \underline{y})$ needed to finance the the maximum level of consumption forever which bounds the sovereign's saving.

The sovereign can decide to default on its debt obligation. We assume in order to avoid cumbersome additional notation that in this case the sovereign is restricted to live in autarky forever.[3] The overall beginning of the period value after imposing the Markovian requirement that only payoff relevant variables are included as state variables is given by

$$V(y, b) = \max \left\{ V^R(y, b), V^A(y) \right\}, \tag{1}$$

where V^R and V^A corresponds to the value under repayment and default/autarky, respectively. The value under financial autarky is then

$$V^A(y) = u(y) + \beta \, \frac{\int_{\underline{y}}^{\overline{y}} \; u(y') \; dF_y}{1 - \beta} = u(y) + \beta \overline{v}.$$

Conversely, if the sovereign decides to repay, it gets access to the market and can issue new long-term debt. The value under repayment reads

$$V^R(y, b) = \max_{b'} \left\{ u \left(y - b + q(b') \; \left[b' - (1 - \lambda) b \right] \right) + \beta \int_{\underline{y}}^{\overline{y}} V(y', b') \; dF_y \right\}. \tag{2}$$

Here, $q(b')$ is the price of one unit of debt when total debt is b'. The assumption of $i.i.d.$ shocks and the restriction to Markovian equilibria ensure that the only argument is the total amount of debt held next period. The solution to this maximization problem gives the policy function $b' = h(y, b)$. In addition,

process is immediate.

[3]It is trivial to extend the defaulted state to a standard one where there is saving after the default and only temporary exclusion from borrowing as is standard in the literature.

we can define the default threshold as

$$d(b) = \min\Big\{\{y : V^R(y,b) \geq V^A(y)\} \cup \{\overline{y}\}\Big\}. \tag{3}$$

This definition says that, for every debt level b, there is either a value of income equating repayment and default, or there is no such value and $d(b) = \overline{y}$. The *i.i.d.* assumption on the endowment process leads to the default threshold being independent of income establishing that debt prices only depend on the total amount of debt.

Additionally, define the debt threshold b^{**} such that $V^R(\overline{y}, b^{**}) = V^A(\overline{y})$ and the *current period* risk-free borrowing limit b^* such that $V^R(\underline{y}, b^*) = V^A(\underline{y})$. Finally, define the repayment surplus as

$$S(y,b) = V^R(y,b) - V^A(y),$$

and the expected continuation surplus as

$$Z(b) = \int_{\underline{y}}^{\overline{y}} V(y,b) - V^A(y)\ dF_y = \int_{\underline{y}}^{\overline{y}} \max\{S(y,b), 0\}\ dF_y.$$

2.2 Equilibrium and Price

Because lenders are risk neutral and competitive, they break even in expectation. Hence, the price of one unit of bond is

$$q(b') = \overline{p} \int_{\underline{y}}^{\overline{y}} \mathbb{I}_{\left\{V^R(y',b') > V^A(y')\right\}} \Big[1 + (1-\lambda)\ q(b'')\Big]\ dF_y,$$

where $\mathbb{I}_{\left\{V^R(y',b') > V^A(y')\right\}}$ is the indicator function that takes value one if $V^R(y',b') > V^A(y')$ and zero otherwise which confirms that the price only depends on b'. Given $d(b)$ and $h(y,b)$, the above expression simplifies to

$$q(b') = \overline{p}\ \Big[1 - F_y(d(b'))\Big] + \overline{p}\ (1-\lambda) \int_{d(b')}^{\overline{y}} q\left[h(y',b')\right]\ dF_y. \tag{4}$$

Since the debt repayment is laddered over multiple periods, the bond price accounts for the sovereign's future actions. Particularly, changes in the price due to $d(b')$ reflect default risk, while those due to $h(y',b')$ reflect dilution risk. Dilution refers to the fact that the sovereign cannot commit not to borrow more in the future. In other words, it cannot commit to maintain a constant default risk until maturity.

Given the definition of the bond price, we can define a Markov competitive equilibrium in this environment.

Definition 1. *A Markov competitive equilibrium in this environment consists of policy functions for the sovereign's consumption, $c(y, b)$, bond holdings, $h(y, b)$ and default, $d(b)$ as well as a bond price schedule $q(b')$ such that*

1. *Taking $q(b')$ as given, $c(y, b)$, $b' = h(y, b)$ and $d(b)$ solve the sovereign's repayment problem in (1)-(3).*
2. *$q(b')$ satisfies (4), meaning that it correctly reflects the default probability and is consistent with zero expected profit.*

The bond price admits a number of kinks or jumps which affect the bond policy function. To better understand this, we can further restrict the level of borrowing by defining the *all time* risk-free debt limit $b^r \leq b^*$ as the largest level of borrowing where the sovereign receives the risk-free rate on debt. Then for any $b' \in [\underline{b}, b^r]$ we have the dilution risk-free price:

$$q(b') = \frac{\overline{p}}{1 - \overline{p}(1-\lambda)} = \frac{1}{r + \lambda}.$$

So we can partition the space of borrowing into:

$$[\underline{b}, \overline{b}] = [\underline{b}, b^r] \cup (b^r, b^*] \cup (b^*, \overline{b}],$$

where the first segment has neither default, nor dilution risk, the second has dilution risk, but no default risk and the final has both default and dilution risk. This suggests there may be multiple points of non-differentiability in the bond price function $q(b)$ if $b^r \neq b^*$.

The following proposition addresses these issues by showing that either the borrower enters the risk borrowing region in which case $b^r = 0$, or it never enters in which case $b^r = b^*$.

Proposition 1 (Two Types of Equilibria)**.** *With low enough income, and outstanding debt $b^* > 0$, either (i) the sovereign finds it optimal to enter the risky borrowing region, i.e. $h(\underline{y}, b^*) > b^*$, in which case $b^r = 0$, or (ii) the sovereign never enters the risky borrowing region $h(\underline{y}, b^*) = b^*$, in which case $b^r = b^*$.*

Proposition 1 establishes that there are two types of behavior in the decision problem. In the first, the sovereign chooses to never exceed the risky debt limit, and so does not face a discount for dilution risk. In the second, the sovereign enters the risky borrowing region with positive probability, so all debt levels have a discount for dilution risk.[4]

The points of non-differentiability b^r and b^* have consequences for the optimal borrowing choice. To the left of $b = b^r$, the sovereign is saving at rate $\frac{1}{r+\lambda}$ and to the right the sovereign is issuing debt at $q \leq \frac{1}{r+\lambda}$ which is either constant or decreasing in $b' \in [0, b^*]$ depending on whether there is default risk – as shown in Proposition 1. This generates a jump in the bond policy function whose size is determined by the decay rate λ. In the limit, as we return to the short-term debt case, $\lambda \to 1$, the jump disappears.

Another possible jump in the bond policy function is at the risk-free borrowing limit, b^*, the consequence of the sign reversal of $h(y, b) - (1 - \lambda)b$. Levels of debt slightly smaller or larger than b^* have different impact on the sovereign's marginal revenue. Particularly, we distinguish 3 cases:

1. Consider the case when $h(y, b)$ approaches b^* and $b^* > (1 - \lambda)b$. In this situation the sovereign is issuing new debt and will choose to stay at b^* for a while rather than incurring the reduction of the price of debt issuance that occurs when the default becomes possible.

2. Alternatively, consider the case where as $(1 - \lambda)b$ approaches b^* we have that $h(b, y) < b^*$. In this case the sovereign is purchasing some of its outstanding debt. Purchasing a little bit more will imply a sudden reduction in its price, inducing the sovereign to jump in its choice and buy even more.

3. Finally, if $b^* = h[(1 - \lambda)b^*, y]$ then the sovereign experiences neither a capital gain nor a capital loss. There is therefore no value of staying at or jumping over b^* and the function is indeed continuous.

The jump in the bond policy reflects the kink in the price function $q(b')$ at $b' = b^*$ whereby the derivative of $q(b')$ drops discretely. In situations where choosing $b' = b^*$ means the agent is buying back debt, that is $b' - (1 - \lambda)b < 0$, there is therefore a discrete upward shift in the valuation of the marginal utility to borrowing as represented by the term $q_b(b')(b' - (1 - \lambda)b)$ in the first-order condition. Intuitively, the sudden faster reduction in the price of debt at this b' raises the marginal value of being in debt above b^* since it makes it cheaper to buy it back.

In what follows, we get rid of these points of non-differentiability in the bond price. In Section 3, we introduce utility shocks to eliminate the threshold b^*.

[4] A natural question to ask is whether these are unique equilibria for economies with different primitives, or whether there is multiplicity and these two types coexist. Aguiar and Amador (2020) have argued the latter is true in a closely related model. Below we show that, by restricting ourselves to equilbria that are the limit of those of economies with finite horizons, equilibria are unique.

3 The Model with Extreme Value Shocks

We introduce extreme value shocks to the value of defaulting and not defaulting. More precisely, the sovereign receives additive utility shocks $\epsilon = (\epsilon^R, \epsilon^A)$, after the endowment y is realized, to the value of repayment and default, respectively. Assume these taste shocks are Type 1 Extreme Value (i.e. Gumbel). Furthermore, the shocks have the property that the expected value of the maximum of the two shocks is zero i.e. the location parameter is $-\alpha(\gamma + \ln 2)$, where $\gamma \approx 0.57$ is the Euler-Mascheroni's constant, and α is the scale parameter.[5] The beginning of the period's value is given by

$$V(y, b, \epsilon) = \max \left\{ V^R(y, b) + \epsilon^R, V^A(y) + \epsilon^A \right\}.$$

Given the assumed distribution of (ϵ^R, ϵ^A), the <u>ex-post</u> probability of repayment, $\phi(y, b)$, is given by the logit form

$$\phi(y, b) = \frac{e^{V^R(y,b)/\alpha}}{e^{V^R(y,b)/\alpha} + e^{V^A(y)/\alpha}}. \tag{5}$$

This is the key advantage of using extreme value shocks. Previously, the repayment decision was binary, while here it is a continuous probability. This eliminates the two thresholds b^* and b^{**} and therefore removes one source of non-differentiability in the bond price. Given this, we can re-partition the space of borrowing into:

$$[\underline{b}, \overline{b}] = [\underline{b}, b^r] \cup (b^r, \overline{b}], \tag{6}$$

Further, note that under the extreme value assumption, we can define

$$G(y, b) = \int_\epsilon V(y, b, \epsilon) \; dF_\epsilon = \alpha \ln \left(e^{V^R(y,b)/\alpha} + e^{V^A(y)/\alpha} \right) - \alpha \ln(2).$$

Note that because there are two sets of random variables, the endowment and the extreme value shocks, we index the probability measures by the variables that they refer to, F_ϵ or F_y. Moreover we have,[6]

$$\phi(y, b) = G_R(y, b).$$

Where G_R is the derivative of G with respect to the value V^R. In addition, define $\chi(b) = \int_y \phi(y, b) \; dF_y$ as the <u>ex-ante</u> repayment probability, and $W(b) = \int_y G(y, b) \; dF_y$ as the expected continuation value

[5] This choice of mean ensures that the option to default does not yield any ex-ante utility or disutility.

[6] This is true for a more general class of taste shocks than Gumbel. See Rust (1988).

for a sovereign in good standing. The values under financial autarky and under repayment as well as the repayment surplus are the same as in Section 2. Also, $Z(b) = W(b) - \int_y V^A(y)\, dF_y$. Finally, given the definition of the repayment probability, the price of one unit of bond reads

$$q(b') = \overline{p}\ \chi(b') + \overline{p}(1-\lambda) \int_{\underline{y}}^{\overline{y}} \phi(y', b')\ q\big[h(y', b')\big]\ dF_y. \tag{7}$$

Note that, with arbitrary low values of α, the problem derived in this section converges to the one posed in Section 2:

Proposition 2 (Zero Extreme Value Shock)**.** *As $\alpha \to 0$, the decision problem converges to the one presented in Section 2.*

4 The Limit of Finite Horizon Economies

We turn to characterize the sovereign's problem under a finite horizon (Section 4.1) and take limits as the horizon becomes infinity (Section 4.2). This will reward us with two results: existence and uniqueness of the equilibrium.

4.1 Decision under Finite Horizon

Consider an environment like that in Section 3 with the exception that the sovereign and the competitive lenders trade long-term debt contracts for a finite number of $T < \infty$ periods which implies that in period T there is no trade in bonds.

Given this, in period T, the beginning of the period's value is given by

$$V_T(y, b, \epsilon) = \max \left\{ u(y-b) + \epsilon^R, u(y) + \epsilon^A \right\}.$$

The borrower does not necessarily defaults on all its debt. This depends on the exact realization of $\left(\epsilon^R, \epsilon^A\right)$. Conversely, in period $T-1$, the value of repayment is given by

$$V^R_{T-1}(y, b) = \max_{b'} \left\{ u\left(y - b + q_{T-1}(b')\ \left[b' - (1-\lambda)b\right]\right) + \beta \int_{\underline{y}}^{\overline{y}} \int_{\epsilon} V_T(y', b', \epsilon')\ dF_\epsilon(\epsilon')\ dF_y(y') \right\},$$

and the value of autarky reads

$$V^A_{T-1}(y) = u(y) + \beta \int_{\underline{y}}^{\overline{y}} V^A_T(y') dF_y(y'),$$

where $V_T^A(y') = u(y')$. From this, the definition of the values in period $t < T-1$ is straightforward.

Bonds depreciate at rate λ and pay a coupon of one unit per outstanding unit of debt until time T. This means that, at time $T-1$, all bonds issued are one period. Thus, the bond price for $b \leq 0$ in $T-1$ is given by

$$q_{T-1}(b) = \bar{p},$$

while for $t < T-1$,

$$q_t(b) = \bar{p} \sum_{k=0}^{T-t-1} (\bar{p}(1-\lambda))^k = q_{t+1}(b) + (\bar{p}(1-\lambda))^{T-t+1}.$$

That is the price of saving is increasing in the number of periods the bond will be paid out. In the limit, the price of saving is $\frac{1}{r+\lambda}$. Conversely, for $b > 0$, in period $T-1$, the bond price is given by

$$q_{T-1}(b) = \overline{p}\ \chi_T(b),$$

which corresponds to the standard pricing formula for a one-period bond with the ex-ante repayment probability defined as $\chi_t(b) = \int \phi_t(y,b) dF_y(y) = \int \frac{e^{V_t^R(y,b)/\alpha}}{e^{V_t^R(y,b)/\alpha} + e^{V_t^A(y)/\alpha}} dF_y(y)$. For any $t < T-1$, then

$$q_t(b) = \overline{p}\ \chi_{t+1}(b) + \overline{p}(1-\lambda) \int_{\underline{y}}^{\overline{y}} \phi_{t+1}(y',b')\ q_{t+1}\left[h_{t+1}(y',b')\right]\ dF_y.$$

By comparing the bond price, the bond policy and the different values over time, there are a number of properties we would like to establish for general t and $t+1$. These properties should ultimately show the sequence of policies, value functions, and prices have a unique limit. This is the purpose of the next section.

4.2 Monotonicity and Limit of Finite Horizon

To later show that the limit of the decision problem as $T \to \infty$ exists and is unique, we first need to show that the main value functions and prices are monotonic over y, b and t:

Proposition 3 (Monotonicity)**.** *In the above environment with $T < \infty$,*

1. **Values and Decision Rules across States** *For all t,*

   1.1 $V_t^R(y,b)$ is strictly increasing in y and strictly decreasing in b.

   1.2 $W_t(b)$ is non-increasing in b.

1.3 $\chi_t(b)$ is non-increasing in b.

1.4 If $q_t(b')$ is non-increasing in b', $h_t(y, b)$ is non-increasing in y and non-decreasing in b.

2. **Debt Prices across Sates:** *For all t, $q_t(b)$ is non-increasing in b.*

3. **Values and Default over Time:** *If $q_t(b) \geq q_{t+1}(b)$, for all $t < T$ and all (y, b),*

   3.1 $S_t(y, b) \geq S_{t+1}(y, b)$.

   3.2 $Z_t(b) \geq Z_{t+1}(b)$.

   3.3 $\chi_t(b) \geq \chi_{t+1}(b)$.

4. **Debt Prices over Time:** *For all $t < T$, $q_t(b) \geq q_{t+1}(b)$.*

The first two parts of the proposition show monotonicity of the value of repayment in (y, b) and of the continuation value, the repayment probability and the bond price in b for a given t. The proofs follow standard arguments. However, one cannot say anything about the monotonicity of consumption. The first reason is that $h(y, b) - (1 - \lambda)b$ can be either positive (i.e. debt accumulation) or negative (i.e. debt buyback). The second reason is that the budget set is not necessarily convex as we see in the next section.

The third part of the proposition states that, under the assumption that the borrowing terms improve as the horizon gets further away, the surplus of repayment and the continuation surplus are increasing, whereas the default probability is decreasing as one approaches the beginning of time in a given state (y, b).[7] However, we cannot say anything about the monotonicity of consumption and the bond policy in t. These results naturally follow from the fact that better prices and a longer horizon (which increases the option value of borrowing) improve the state of the sovereign inducing larger values.

The last part of the proposition states that the borrowing terms are indeed worsening as one approaches T. This follows from fact that the repayment surplus shrinks because the opportunities for further borrowing are decreasing. While intuitive, this result is not trivial to prove. The default risk for any state (b, y) is decreasing with the horizon of the problem, whereas the part of the debt price dealing with dilution risk is not obvious. We cannot say anything about the monotonicity of the borrowing policy h_t in time, as borrowing may increase or decrease in response to better prices $q_t \geq q_{t+1}$. Instead, we prove the choices of bonds leading to higher dilution risk are not optimal. Intuitively, for a given state

[7] It can be shown that $V_t^R(y, b) > V_{t+1}^R(y, b)$ and $V_t^A(y) > V_{t+1}^A(y)$ for all $t < T$ and all (y, b) under the additional assumption that $u(\underline{c}) > 0$ where $\underline{c}$ is the lowest level of consumption achieved in equilibrium by adding a suitable large positive constant to the utility function.

(y, b) it is never optimal for a borrower to move farther up the borrowing Laffer curve at date t, than at date $t+1$. Doing so leads to too much consumption today, and the borrower could reduce their borrowing and be better off. This shows that $q_{t+1}(b) > q_t(b)$ as a result of both lower default, and lower dilution risk as the horizon increases.

Note that proposition 3 holds without taste shocks. All the proofs apply to the decision problem in Section 2 upon suitable re-definition of values and the default set. As we will see in the next section when we derive the GEE, the taste shocks ensure convexity of the decision problem and differentiability of the main policy functions. They are not necessary for the monotonicity of the price, the value functions or the repayment probability.

It is straightforward to show that, for a given price function $q(b')$ the sovereign's problem is a contraction, and the problem is continuous in $q(b')$. This implies that if $q_t(b') \to q^*(b')$ as the horizon of the problem increases to infinity, then the solution to the limiting economy is simply the sovereign's problem with the price taken to be $q^*(b')$. So the question becomes does such a limiting price $q^*(b')$ exist for the finite horizon problem?

The answer is affirmative if $q_t \geq q_{t+1}$ as this is a sequence of nonincreasing (in b') and bounded functions, it must converge uniformly to some limiting q^*. Hence, given Proposition 3, we can show that the limit of the decision problem under finite horizon exists and is unique through the fixed point theorem of monotone operators in Stokey et al. (1989). Given this definition and the monotonicity of the repayment surplus $S(y, b)$ shown in Proposition 3, we come up with the following proposition

Proposition 4 (Existence and Uniqueness)**.** *Let q^* be the limiting price function of the finite horizon problem, and $\Omega = [\underline{y}, \overline{y}] \times [\underline{b}, \overline{b}]$ be the state space. For a function $S : \Omega \to \mathbb{R}$ defined as $S(y, b) = V^R(y, b) - V^A(y)$, define the operator $(KS)(\Omega; q^*)$ as:*

$$(KS)(y, b; q^*) = \sup_{b'} \left\{ u(y - b + q^*(b')[b' - (1-\lambda)b]) - u(y) + \beta \int_{\underline{y}}^{\overline{y}} \max\left\{ S\left(y', b'; q^*\right), 0\right\} dF_y\left(y'\right) \right\}.$$

Then (i) the limit of the finite horizon problem is the fixed point of K given by $\overline{S}(y, b; q^*)$, and (ii) this fixed point exists and is unique.

Proposition 4 states that in this environment, there is a unique equilibrium that is the limit of the equilibrium of economies with finite horizons. This unique equilibrium is a Markov equilibrium consistent with Definition 1. In other words, we rule out Markov-like equilibria that could be sustained by means of trigger strategies. In light of this, it seems that the multiplicity uncovered by Aguiar and Amador

(2020) might be due to all but one of those equilibria not being the limit of those of finite economies.

5 The Generalized Euler Equation

Having shown existence and uniqueness, we now characterize the decision problem through the generalized Euler Equation (GEE), that is the Euler equation which includes the derivative of bond prices, and through them the derivatives of future actions with respect to current actions.

5.1 Obtaining the Generalized Euler Equation

The GEE provides an important analytical tool to characterize the decision of the sovereign in cases where default occurs (in economies where there is no default the standard Euler equation applies). However, its derivation requires the bond price $q(b)$, the bond policy $h(y, b)$, and the repayment probability $\phi(y, b)$ to be differentiable in b. At this stage, we proceed by assuming that they are indeed differentiable in the interior of the two regions in (6) and we will turn to prove it below in Section 5.2.

Besides differentiability, the statement that the GEE is a sufficient condition for optimality requires that the decision problem is convex. This is important as Chatterjee and Eyigungor (2012) show that the budget set under repayment is not necessarily convex. Without convexity, the first-order conditions are necessary but not sufficient. Concretely, there might be multiple local maxima in the GEE which difficults the use of such analytical tool to characterize the equilibrium outcome. Hence, we assume that the scale parameter α is sufficiently large to ensure convexity of the problem. We discuss this assumption in Section 5.2. If the scale parameter α is not sufficiently large to ensure convexity the GEE still exists but will typically have more than one solution only one of which will be the global optimum. The GEE can still be used to characterize (and compute) the solution but we also have to keep track of the implied values to find out which of the solutions yields the global maximum.

Given the convexity of the problem and the differentiability of $q(b)$, $h(y, b)$ and $\phi(y, b)$, we can take the first-order condition of Equation (2) with respect to b',

$$u_c(\mathcal{C}(y, b, b'|q)) \left[q(b') + q_b(b')\ (b' - (1-\lambda)b)\right] = \\ \beta \int_{\underline{y}}^{\overline{y}} \phi(y', b')\ u_c(\mathcal{C}(y', b', h(y', b')|q))\ [1 + (1-\lambda)\ q(h(y', b'))]\ dF_y,$$

where $\mathcal{C}(y, b, b'|q) \equiv y - b + q(b')(b' - (1-\lambda)b)$ corresponds to consumption today and $\mathcal{C}(y', b', h(y', b')|q)$ to consumption tomorrow. This notation makes explicit the functional dependencies. In addition, $q_b(b')$ is the derivative of the bond price with respect to b'. The left-hand side of the above expression represents the marginal benefit of one additional unit of debt while the right-hand side corresponds to the marginal cost. The marginal benefit is the consumption gain from marginal borrowing taking into account the impact it has on the price of the debt. The marginal cost of an additional unit of borrowing is the expected marginal utility loss of paying the coupon and rolling over unmatured debt at tomorrow's price in repayment states.

For a given state (y, b) and borrowing b', the first-order condition contains the functions ϕ, h, q and q_b. Since we seek a GEE that involves decision rules only, and not equilibrium price functions, one step is the substitution of the price derivative q_b. Taking the first derivative of the price of debt as described in Equation (7) gives

$$\begin{aligned} q_b(b') =& \overline{p}(1-\lambda) \int_{\underline{y}}^{\overline{y}} \phi(y', b')\ q_b \left[h(y', b')\right] h_b(y', b')\ dF_y \\ &+ \overline{p} \int_{\underline{y}}^{\overline{y}} \phi_b(y', b') \left[1 + (1-\lambda)\ q(h(y', b'))\right]\ dF_y. \end{aligned} \tag{8}$$

where $h_b(y', b')$ and $\phi_b(y', b')$ denote the first derivative of the bond policy and the default function, respectively. The first term with the integral is the dilution risk, while the second term gives the loss of value per unit of debt weighted by the marginal probability of default. The dilution term of the price derivative is therefore itself represented by the future price derivatives $q_b(h(y', b'))$. In fact, we can use the value of q_b implied by the first-order condition to get an expression that does not depend on future derivatives. Inverting the first-order condition, in an equilibrium where $b' = h(y, b)$, we can write the price derivatives as $q_b(h(y, b)) = \mathcal{B}(y, b|h, \phi, q)$ where $\mathcal{B}$ is a known expression given by

$$\mathcal{B}(y, b|h, \phi, q) \ = \ \frac{\int_{\underline{y}}^{\overline{y}} \phi(y', h(y, b))\ u_c(c')\ [1 + (1-\lambda)q(h(y', h(y, b)))]\ dF - u_c(c)q(h(y, b))}{u_c(c)\ [h(y, b) - (1-\lambda)b]}, \tag{9}$$

with $c = \mathcal{C}(y, b, h(y, b)|q)$ and $c' = \mathcal{C}[y', h(y, b), h(y', h(y, b))|q]$.

Since the present self assumes equilibrium in future, the future derivatives can analogously be expressed so that $q_b(h(y', b')) = \mathcal{B}(y', b'|h, \phi, q)$, and the current price derivative, as a function of b', becomes

$$\begin{aligned} q_b(b') = \overline{p}(1-\lambda) \int_{\underline{y}}^{\overline{y}} \phi(y', b')\ \mathcal{B}(y', b'|h, \phi, q)\ h_b(y', b')\ dF_y& \\ + \overline{p} \int_{\underline{y}}^{\overline{y}} \phi_b(y', b') \left[1 + (1-\lambda)\ q(h(y', b'))\right]\ dF_y&. \end{aligned}$$

Combining the above expressions gives the GEE. Formally,

$$u_c(c)\Bigg[q(b') + (b' - (1-\lambda)b)\ \Bigg\{\overline{p}(1-\lambda)\int_{\underline{y}}^{\overline{y}} \phi(y', b')\ \mathcal{B}(y', b'|h, \phi, q)\ h_b(y', b')\ dF_y$$
$$+\,\overline{p}\int_{\underline{y}}^{\overline{y}} \phi_b(y', b')\ [1 + (1-\lambda)\ q(h(y', b'))]\ dF_y\Bigg\}\Bigg]$$
$$= \beta\int_{\underline{y}}^{\overline{y}} \phi(y', b')\ u_c(c')\ [1 + (1-\lambda)\ q(h(y', b'))]\ dF_y. \quad \text{(GEE)}$$

with $c = \mathcal{C}(y, b, h(y, b)|q)$ and $c' = \mathcal{C}[y', h(y, b), h(y', h(y, b))|q]$.

The (GEE) still contains the price function q. Yet Equation (7) fully characterises that price function q for given decision rules ϕ and h. Note then how substituting q from (GEE) yields a formula that is only a function of current and future decision rules without any need of involving markets. It is then a characterization that only involves the game against future selves, albeit all future decision rules, so using the pricing function q provides a convenient way to simplify notation.

Based on our previous discussion around (6), there are two distinct regions for the solution of b'. First, in points such that $b' < 0$, there is no default risk as the sovereign can only repudiate debt (i.e. $b' > 0$). Thus, in such points, the optimal borrowing is the solution to the standard Euler equation

$$u_c(c)\ q(b') = \beta\int_{\underline{y}}^{\overline{y}}\ u_c(c')\ [1 + (1-\lambda)\ q(h(y', b'))]\ \ dF_y. \quad \text{(EE)}$$

with $c = \mathcal{C}(y, b, h(y, b)|q)$ and $c' = \mathcal{C}[y', h(y, b), h(y', h(y, b))|q]$.

Second, in points such that $b' > 0$, there is both dilution and default risk meaning that the optimal borrowing is the solution to (GEE). The GEE therefore enables a complete characterization of the sovereign's interior borrowing choice. When $b' > 0$, the borrowing policy solves (GEE) and there is both dilution and default risk. When $b' < 0$, the borrowing policy solves the standard Euler equation (EE). If none of those conditions are met, then the borrowing is a corner solution at $b' = 0$.[8]

[8]Without extreme value shocks, we have three regions given the threshold value b^*. More precisely, in points such that $b' < 0$, the optimal borrowing is the solution to the EE. In points such that $b' \in (0, b^*)$, the optimal borrowing is the solution to the GEE with dilution risk and without default risk. Finally, in points such that $b' > b^*$, the optimal borrowing is the solution to the GEE with both dilution and default risk. There are further two corner solution, $b' = 0$ and $b' = b^*$ where the GEE is not satisfied.

5.2 Convexity and Differentiability

The use of the GEE as a sufficient condition to characterize optimality critically depends on two elements: the convexity of the decision problem and the differentiability of q, ϕ and h. We first investigate the former.

The first-order conditions are sufficient for a maximum under a concave objective function and a convex budget set. The introduction of extreme value shocks addresses the issue of non-convexities as it allows agents to randomize over decisions. In each $b > 0$, there is always a positive probability that the default option is chosen. Such probability distribution can convexify the decision problem. Nevertheless, the mere introduction of randomization is usually not sufficient and one needs to ensure that randomization is intense enough.[9]

The size of the scale parameter α is therefore critical in ensuring convexity. As $\alpha \to 0$, we return to the case studied in Section 2. That is, the probability of defaulting is becoming degenerate. As a result, small changes in the bond prices can lead to large changes in the borrowing decision. In other words, as α gets closer to zero, randomization vanishes and non convexities prevent the use of the GEE. In contrast, as $\alpha \to \infty$, then the probability of defaulting approaches 0.5 in every state in which $b > 0$. In this case, the decision problem becomes convex as we show in the following proposition.

Proposition 5 (Convexity)**.** *If α is sufficiently large, then the decision problem is convex.*

Regarding differentiability, as already said, there are two forms of discontinuity: kinks and jumps. Kinks are not an issue as integration over kinks preserve not only continuity but also differentiability. In contrast, jump discontinuities are problematic as the integral is continuous but not differentiable at this point. We got rid of potential jumps at b^* with the continuous repayment probability obtained with extreme value shocks. We further rule out jumps at zero debt by restricting our attention to the case in which $b \geq 0$.

With long-term debt, the bond price depends on the optimal policy h, so differentiability of the bond price has to be established recursively through the finite horizon problem presented in Section 4.

Proposition 6 (Smoothness)**.** *If $b \geq 0$, ϕ, h and q are of class $\mathcal{C}^\infty$ almost everywhere.*

The proposition relies on an inductive argument. At time T, the bond policy function is a constant. Hence, it is obviously of class $\mathcal{C}^\infty$. Given this, at time $T-1$, we can show that the GEE is itself of class $\mathcal{C}^\infty$ except for a discontinuity at zero. By application of the implicit function theorem, the bond

[9] See also the discussion in Iskhakov et al. (2017).

policy at $T-1$ inherits the properties of the GEE. That is, it is smooth except for a discontinuity at zero. Given this property of the bond policy, we can show that the bond price at $T-2$ is also of class $\mathcal{C}^\infty$ almost everywhere. We then repeat this argument backward until $t=0$.

The fact that we rely on $\mathcal{C}^\infty$ is crucial as the proof works recursively through the finite horizon problem. If for some $0 < t < T$, the bond policy function h_t were of class $\mathcal{C}^k$ for $k < \infty$, then h_{t-1} would be of class $\mathcal{C}^{k-1}$ since the GEE at time $t-1$ depends on $h_{b,t}$ which is of class $\mathcal{C}^{k-1}$. Hence, if $k < \infty$, one loses one degree of differentiability for each iteration. This could eventually prevent to show that ϕ, h and q are differentiable.

Once the differentiability of ϕ, h and q has been established, we need to show that the derivatives converge to the appropriate policies.

Proposition 7 (Limiting Bond Policy)**.** *The bond policy function, h_t, converges pointwise to a function h and the derivative of the bond policy function, $h_{b,t}$, converges pointwise to $h_b = \frac{d}{db}h$.*

For the bond policy function, the argument is the following. On the one hand, the application of the implicit function theorem enables to characterize the bond policy, h, directly from the GEE. Especially, it states where for a given t there is a unique implicit function, h_t, of the same differentiability class as of the GEE at t. Moreover, real limits are unique when they exist meaning that $h_{t,b} = \frac{d}{db}h_t$ is unique as well. On the other hand, the GEE evaluated at the optimal borrowing converges pointwise to zero in every t. This gives sequences of $\{h_n\}$ and $\{h_{b,n}\}$ which converge by construction.

Given Proposition 7, we can additionally show the convergence of the bond price. The argument follows from the fact that given h_n and $h_{b,n}$, the bond price and its derivative are a contraction.

Proposition 8 (Limiting Bond Price)**.** *Let $\{q_n\}$ and $\{q_{b,n}\}$ denote the sequence of bond price and its derivative. q_n converges uniformly to q^* and $q_{b,n}$ converges uniformly to q^*_b, implying that $q^*_b = \frac{d}{db}q^*$.*

6 Computation

We now describe briefly the advantages that our characterization of the GEE provides when it turns to solve the model computationally. There are three reasons that give an advantage. The use of policy function iterations because we can use first order conditions, the natural way that this allows us to interpolate in between states and actions and, for the particular case of risk aversion of two, the fact that the use of the endogenous grid method allows to avoid the use of root finding to solve a nonlinear equation. This is because, given the lack of characterizations of the first order conditions in the literature, this model is almost always solved via discretization of the state and action space using value function iteration methods.[10]

6.1 Methods

The solution method commonly referred to as value function iteration (VFI) uses the following set of functional equations

$$h(b,y) = \arg\max_{b'} \left\{ u\left[y - b + q(b')\ (1-(1-\lambda)b)\right] + \beta\ W\left(b'\right) \right\}, \tag{10}$$

$$V^R(b,y) = u\left[y - b + q(h(b,y))(h(b,y) - (1-\lambda)b)\right] + \beta\ W\left(h(b,y)\right), \tag{11}$$

$$W(b') = \int \alpha\ \log(e^{V^R(b',y')/\alpha)} + e^{V^A(y')/\alpha})\ dF_y\left(y'\right), \tag{12}$$

$$q\left(b'\right) = \bar{p} \int \phi\left(b',y'\right)\ \left[1 + (1-\lambda)\ q\left(h(b',y')\right)\right]\ dF_y\left(y'\right), \tag{13}$$

$$\phi(b,y) = \frac{e^{V^R(b',y')/\alpha}}{e^{V^R(b',y')/\alpha} + e^{V^A(y')/\alpha}}. \tag{14}$$

The left hand side of this system is also the set of unknown functions, $\{h, V^R, W, q, \phi\}$. This is solved by discretizing both the state and the action space transforming this problem in one of finite dimension, and it is typically solved by iterations from the future to the present although it is not necessary. Yet, accuracy requires a large number of points in the grid for debt holding to avoid the solutions being the result of rounding rather than of maximizing.

Methods described as policy function iteration (PI) use first order conditions, which in our case imply the GEE. They do not typically operate via discretizing the state or the choice set. These methods use Equations (11) to (14) but not Equation (10) and add four additional functional equations:

$$\begin{aligned} 0 = u_c\left[y - b + q(h(b,y))(1-(1-\lambda)b)\right]\left[q\left(h(b,y)\right) + \right. \\ \left. q_b\left[h(b,y)\right]\left[h(b,y) - (1-\lambda)b\right]\right] + \beta W_b\left[h(b,y)\right], \end{aligned} \tag{15}$$

$$W_b(b) = -\int \phi(b,y)\ u_c\left[y - b + q(h(b,y))(1-(1-\lambda)b)\right]\ dF_y(y), \tag{16}$$

[10] There are exceptions. Jang and Lee (2021) also use the Endogenous Grid Method for a sovereign default problem with short term debt. They seem to implicitly assume differentiability. Kiiashko and Maliar (2021) in independent preliminary work use the Endogenous Grid Method with long term debt. They add an i.i.d. normal shock to the value of default, and then they compute numerically the derivatives of both the value function and the pricing function, which may solve the problems that we highlight in this paper. There is no information on the extent to which this strategy works. The numerical computation of the derivatives of the expected value function avoids using the information that our characterization of the GEE allows.

$$q_b(b) = \bar{p} \int \phi_b(b,y) \ \{1 + (1-\lambda) q[h(b,y)]\} \ dF_y(y) +$$
$$\bar{p}(1-\lambda) \int \phi(b,y) \ q_b[h(b,y)] \ h_b(b,y) \ dF_y(y), \tag{17}$$
$$\phi_b(b,y) = -\frac{1}{\alpha} u_c \left[y - b + q(h(b,y))(1-(1-\lambda)b)\right]$$
$$\{1 + (1-\lambda) \ q[h(b,y)]\} \ \phi(b,y) \ [1 - \phi(b,y)] . \tag{18}$$

There are four additional unknowns, $\{q_b, W_b, \phi_b, h_b\}$. This makes for a total of 8 functional equations but 9 unknown functions. Of these, the derivative of the borrowing policy h_b has no analytic expression in terms of the remaining equilibrium objects. We can however, approximate h_b using finite differences in particular grid points. That is, for b_i in the grid

$$h_b(b_i, y_j) \approx \frac{h(b_{i+1}, y_j) - h(b_{i-1}, y_j)}{2\Delta_b}, \qquad 1 < i < N_b,$$
$$h_b(b_i, y_j) \approx \frac{h(b_i, y_j) - h(b_{i-1}, y_j)}{\Delta_b}, \qquad i = N_b,$$
$$h_b(b_i, y_j) \approx \frac{h(b_{i+1}, y_j) - h(b_i, y_j)}{\Delta_b}, \qquad i = 1.$$

The monotonicity of the borrowing policy $b' = h(b,y)$ implies that h is invertible in b', i.e. there exists a function $b = g(b', y)$ which provides a theoretical justification for the use of the Endogenous Grid Method in computing the solution to the problem.

In general solving the GEE (Equation (15)) for b' requires a numerical solver. In standard models without time inconsistency, the intertemporal Euler equation can be solved using the endogenous grid method which achieves large speed gains. In our case this is not true in general because of the terms involving derivatives of decision rules. However, in the case of CRRA utility with a risk aversion of 2, we can use it because consumption can be obtained as the solution to a second order polynomial.[11] Policy

[11] Using the budget constraint to solve for b gives $b = \frac{y + q(b')b' - c}{1 + (1-\lambda) q(b')}$, and substituting this expression into the GEE and rearranging yields

$$\beta W_b(b') c^2 + (1-\beta) \frac{q_b(b')(1-\lambda)}{1+(1-\lambda)q(b')} c + (1-\beta) \left\{ \left(q(b') + q_b(b')b' - (1-\lambda) q_b(b') \left[\frac{y + q(b')b'}{1 + (1-\lambda)q(b')} \right] \right) \right\} = 0, \tag{19}$$

which is a quadratic equation for c. The solution is given by

$$c = \frac{-a_2 \pm \sqrt{a_2^2 - 4a_1a_3}}{2a_1},$$
$$a_1 = \beta W_b(b') < 0,$$
$$a_2 = (1-\beta) \frac{q_b(b')(1-\lambda)}{1+(1-\lambda)q(b')} < 0,$$

function iteration methods are also typically solved by iterating from the future (like taking the finite horizon problem up to infinity).

We compare heuristically the computational properties of four methods: (i) Value function iteration without taste shocks, (ii) Value function iteration with taste shocks, (iii) Policy function iteration with taste shocks, and (iv) Policy function iteration using the endogenous grid method with taste shocks. We do so by means of a generic example: Utility is CRRA $u(c) = (1-\beta)\frac{c^{1-\sigma}}{1-\sigma}$ with $\sigma = 2$. The risk free rate is $r = 0.04$. The discount factor is $\beta = 0.92$. The bond maturity parameter is $\lambda = 0.1$. The taste shock parameter is $\alpha = 0.01$. The choice of the taste shock balances a desire for smoothness and concavity (larger shock) with a solution that is close to the model without taste shocks ($\alpha = 0$). Income is discretized into a grid of N_y equally spaced points $\mathcal{Y} = \{y_1, y_2, \dots, y_{N_y}\}$ with corresponding probability mass function π_j. $N_y = 41$ and $y \sim U[1-\gamma, 1+\gamma]$. The parameter γ is chosen based on its relation to the volatility of income $\sigma_y = \sqrt{3}\gamma$, and $\sigma_y = 0.35$.

Both value function iteration methods (with and without extreme value shocks) require discretization of the state and choice spaces. To this end, we set bond holdings to be in an equally spaced grid for borrowing $\{b_1, b_2, \dots, b_{N_b}\}$ with $b_1 = 0$ and $b_{N_y} = 0.25$. To minimize the effect of discretization errors we use a large $N_b = 1500$ and the choice of b' is also forced to lie in the grid. When solving using VFI we use the monotonicity property of the borrowing policy to limit the choice set of b' to decrease computational time. Policy iteration allows for not making choices in the grid, so fewer points are required. For the two policy iteration methods, we set $N_b = 101$. Using values and derivatives of prices q, expected utilities W, saving functions h (recall that for the later we obtain and derivatives by numerical differentiation) we interpolate outside the grid via the construction of piecewise Hermitian cubic polynomials which match the level and first derivative of the function at every grid point. S: Note that the endogenous grid method needs a grid for b and b'. N_b is the number of grid points for b and $N_{b'}$ is the number of grid points for b'. We invert $h(y, b) = b'$ to get $g(y, b') = b$, and solve for b given (b', y) using this relation. This gives a set of b that we interpolate onto the N_b grid. Generally choosing $N_{b'} > N_b$ is necessary for an accurate solution. Thus, we use a grid of $N_{b'} = 500$ equally spaced points

$$a_3 = (1-\beta)\left\{q(b') + q_b(b')b' - (1-\lambda)q_b(b')\left[\frac{y + q(b')b'}{1 + (1-\lambda)q(b')}\right]\right\} > 0.$$

Note there are possibly two roots to this equation, however, we know it is optimal to choose the highest level of c, and given that $\sqrt{a_2^2 - 4a_1a_3} > a_2$ and $a_1 < 0$, the solution must be

$$c = \frac{-a_2 - \sqrt{a_2^2 - 4a_1a_3}}{2a_1}.$$

over the same interval covered by $\mathcal{B}$.

6.2 Accuracy and Speed

Figure 1 displays the convergence properties by showing the errors in successive iterations in the sup norm. The first thing to note is that without taste shocks value function iteration did not converge with the grid chosen. Second, even with taste shocks convergence of value function iteration is relatively poor, its precision (the error in subsequent iterations) is limited to 10^{-3} in price and 10^{-2} in values. Policy iterations either with or without taking advantage of the endogenous grid method are much more precise yielding differences in subsequent iterates that are close to machine precision after 1000 iterations.

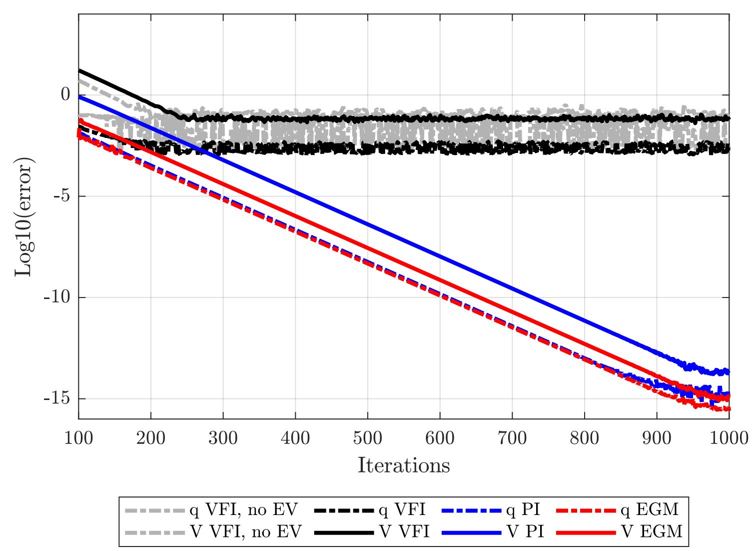

Figure 1: $\log_{10}$ errors of the price function and the value of repayment after successive iterations of each algorithm.

The real advantage of policy iteration methods is computational time. In our example, using value function iteration with(without) extreme value shocks took 5.17(4.45) minutes to make 1000 iterations and the errors stabilized well above standard thresholds for convergence. Using policy iteration with a numerical root finder took 6.32 minutes to make 1000 iterations, and both the price and value functions converged with respect to a tolerance of 10^{-6} in 3.03 minutes. Finally, when the endogenous grid method was used, the computing time dropped to 4.03 seconds to make 1000 iterations, and the price and value functions converged with respect to a tolerance of 10^{-6} in 1.63 seconds, reflecting the additional gains in speed from avoiding numerical root-finding.

6.3 Convexity with Extreme Value Shocks

The use of first order conditions methods like the GEE relies on the concavity of the objective function and convexity of the choice set. Violation of these conditions arise when the sovereign has accumulated large level of debt, and then receives a good endowment realization. In the limiting case with $\alpha = 0$ there is a kink in the Laffer curve at the risk-free debt limit b^*, which leads the possibility of two points where the GEE is satisfied, one below b^* and another above. From Proposition 5 we know that a sufficiently large extreme value shocks convexifies the problem, but it is a quantitative question how large the shock needs to be in practice. Figure 2 shows this possibility of multiple local maxima (with one global maximum) can happen with extreme values shocks when the shock is very small. However, we can see the size of the shock necessary to rule out multiple local solutions is in this case as small as $\alpha = 0.01$ meaning using the GEE is a valid computational approach for a model that is very close to the limiting case $\alpha = 0$.[12]

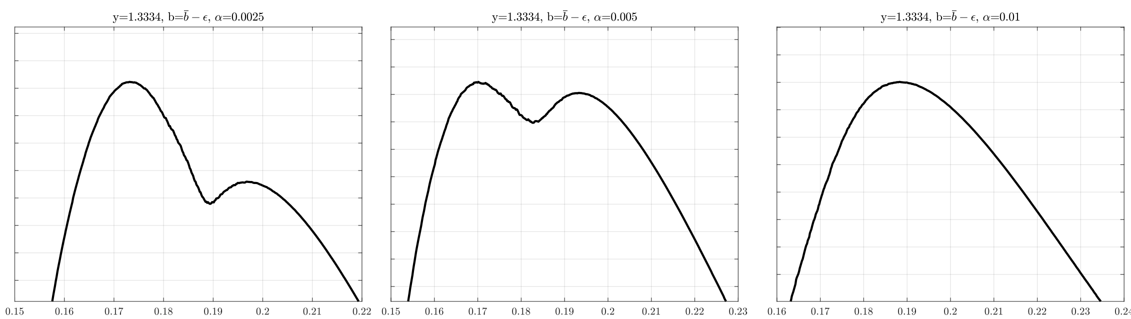

Figure 2: Plots of the objective function $u(c) + \beta W(b')$ as a function of the choice b' for differing levels of taste shocks.

If the scale parameter α is not sufficiently large to ensure convexity the GEE still exists but will typically have more than one solution. With multiple solutions, we need to keep track of the value function to pick the global optimum. Because the multiple solution arises from the possibility of buybacks, there are at most two local maxima. This suggests a simple root-finding algorithm in which we start at low and high guesses of b'. If the two guesses lead to the same solution, we are fine. Otherwise, we simply pick the better one with the value function. With more than two local maxima, we need to take multiple guesses and evaluate them all with the value function.

[12] Figure 2 shows the objective for a particular point in the state space, but we have verified $\alpha = 0.01$ is sufficient to rule out multiple local optima at all points in the state space.

6.4 Decomposing the GEE Numerically

We now describe the behavior of the sovereign and the role of the different terms in the GEE in determining the marginal value of borrowing.

First, Figure 3 shows the decision rules of the sovereign and corresponding debt price at all points in the state space. From the bottom-left panel, we see when the sovereign enters the risky borrowing region they face a sharp decline in the price of their debt. The borrowing policy $b' = h(y, b)$ is monotone in both y and b, and we see when the sovereign has accumulated large levels of debt, a positive endowment realization leads to sharp reductions in borrowing. This can be seen in the ridge that forms in the buyback $b' - (1 - \lambda)b$. This reduction in debt leads to higher debt prices which means the sovereign may optimally choose lower levels of consumption at higher endowment realizations for a given level of debt. We see this non-monotonicity in consumption in the top-left panel of Figure 3. Non monotonic consumption occurs where $b' < (1 - \lambda)b$ when debt is high $b > b^*$, and the policy is discontinuous (without EV shocks) as the borrower exits the borrowing region by jumping down. With EV shocks this is no longer a jump, but a steep slope in the borrowing policy. This means consumption is lower as the borrower is paying a high price to buy back its debt and exits the risky borrowing region.

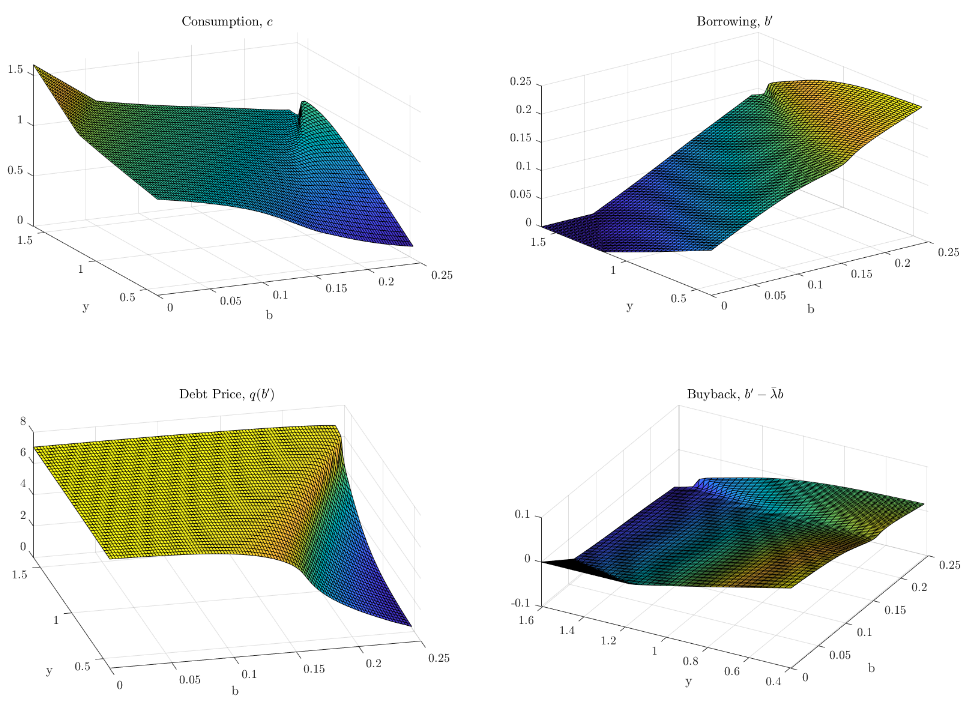

Figure 3: Policy functions evaluated at all points (y, b) in the state space.

To understand the forces that shape the optimal choices of the sovereign, we plot the left-hand side of the GEE in Figure 4. Recall from Equation (8) the derivative of the debt price is the sum of the dilution risk and default risk terms. The top-left panel shows the marginal value of borrowing –the first term of the right hand side of Equation (15)– which is highest near the boundary of the risky-borrowing region. The next three panels decompose the marginal value of borrowing into (i) the marginal utility ignoring the derivative of the debt price $u_c(c)q(b')$, (ii) the marginal utility accounting for default risk $u_c(c)(b' - (1-\lambda)b)q_b^{def}(b')$, and (iii) the marginal utility accounting for dilution risk $u_c(c)(b' - (1-\lambda)b)q_b^{dil}(b')$. First, in the absence of the effects of the price derivative $u_c(c)q(b')$, the marginal utility of consumption is high when debt is high and the endowment is low, and there is incentive for the sovereign to increase borrowing. Second, the effects of the price derivative in terms of default and dilution discipline the sovereign into borrowing less, which we can see in the bottom two panels, where the marginal revenue from addition borrowing becomes negative. Third, the disciplining effects of the price derivative are mainly coming through the discount for default risk. The effects of default risk are at least twice as large as dilution risk when the sovereign is in the risky borrowing region.

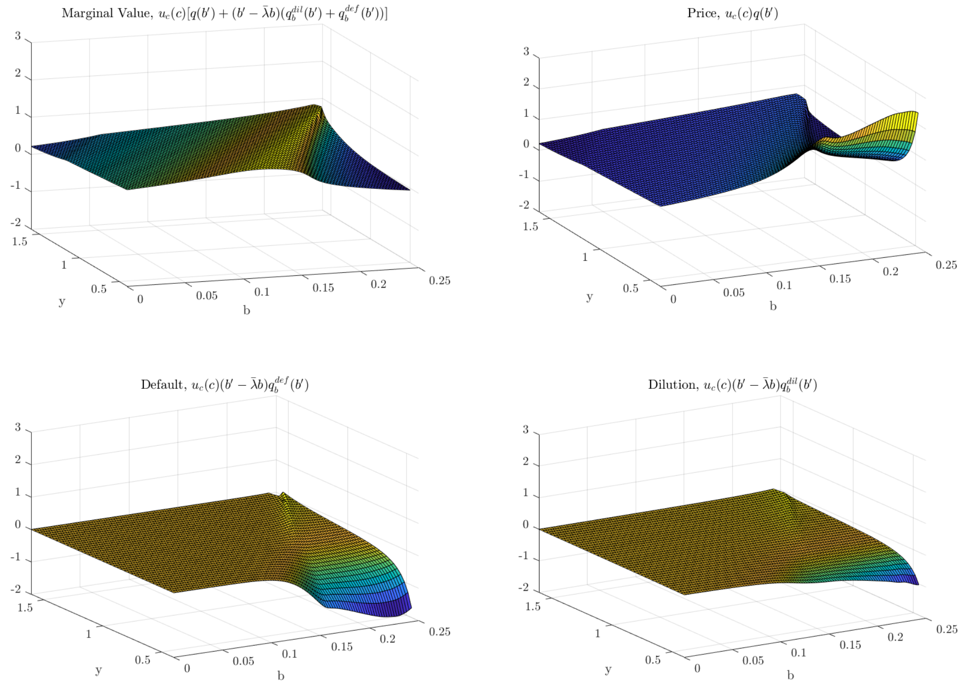

Figure 4: Decomposition of the left hand side of the Generalized Euler Equation into its individual components evaluated at all points (y, b) in the state space.

7 Conclusion

In this paper we have characterized the solution to the sovereign default problem with long term debt. We have used the GEE to describe how default risk and dilution risk shape the decision. Further, we have documented the existence of a kink in the pricing function where default risk starts that results in violation of the GEE to characterize the solution and that, depending on the size of existing debt, may result in the decision being held constant for a variety of states or in a jump of debt issuance. Adding noise to the decisions, in our case in the form of extreme value shocks, eliminates the kinks in the pricing function and convexify the decision problem yielding a GEE with default and dilution risk everywhere to characterize the solution. In addition, we have shown that the equilibrium of finite economies is unique and that it converges to our solution as the number of periods goes to infinite showing uniqueness of equilibria when we require the equilibrium not only to be Markovian but to be the limit of finite economies. Finally we have shown by means of an example the enormous gains in accuracy and speed obtained by using policy function iteration methods that take advantage of the GEE that we have derived. We hope that this paper ends the nature of black box solution methods that are used to deal with economies with sovereign default.

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Appendix: Proofs

Proposition 1 (Two Types of Equilibria)**.** *With low enough income, and outstanding debt $b^* > 0$, either (i) the sovereign finds it optimal to enter the risky borrowing region, i.e. $h(\underline{y}, b^*) > b^*$, in which case $b^r = 0$, or (ii) the sovereign never enters the risky borrowing region $h(\underline{y}, b^*) = b^*$, in which case $b^r = b^*$.*

Proof of Proposition 1. Case (ii) is straightforward given our definition of b^r in Section 2.1. To prove Case (i), we first show that the borrower increases its indebtedness when the income realization is sufficiently low. This is the purpose of the following lemma.

Lemma 1. *For any $0 < b < b^*$ there exists some $\delta(b) > 0$, such that the borrowing policy satisfies $h(x, b) > b$ for each $x \in [\underline{y}, \underline{y} + \delta)$.*

Proof of Lemma 1. The proof builds on the following result: for any $0 < b < b^*$, the optimal consumption choice at the lowest income satisfies $c(\underline{y}, b) > \underline{y}$.

Suppose not, that is $c(\underline{y}, b) \leq \underline{y}$ for $0 < b < b^*$. Then, for some small Δ, the sovereign could issue an additional $\frac{\Delta}{q(b')}$ units of debt today which would yield $u_c(c(\underline{y}, b))\Delta$ utils of benefit. The expected cost of doing so is:

$$\frac{\Delta\beta}{q(b')}\int u_c(c(y', b'))dF_y(y') < u_c(c(\underline{y}, b))\Delta.$$

So the sovereign is better off doing so, a contradiction to optimality. Now, define $\delta(b)$ as:

$$\delta(b) = \sup\{y : c(y, b) \geq y\} - \underline{y}.$$

Given this result, the claim follows from the budget constraint. For any $x \in [\underline{y}, \underline{y} + \delta(b))$,

$$c(x, b) - x = q(b')[b' - (1 - \lambda)b] - b > 0.$$

Rearranging yields to

$$b' > \frac{b}{q(b')} + (1 - \lambda)b.$$

Note, the bond price is bounded above by the dilution risk-free price

$$q(b') \leq \frac{1}{r + \lambda} < \frac{1}{\lambda}.$$

Which implies:

$$b' > \frac{b}{q(b')} + (1-\lambda)b > \lambda b + (1-\lambda)b = b.$$

Proving the original claim. $\square$

Case (i) relies on the result we established in Lemma 1. Note, Lemma 1 implies if the sovereign starts with $b_0 > 0$ and receives a bad shock, $x_1 \in [\underline{y}, \underline{y} + \delta(b_0))$, then debt increases $b_1 > b_0$, and after another bad shock $x_2 \in [\underline{y}, \underline{y} + \delta(b_1))$, then debt increases again $b_2 > b_1$, and so on. So with positive probability, after a sequence of bad shocks, the sovereign will end up at the risk-free limit b^*.
At which point the sovereign will enter the risky borrowing region by assumption if income is low enough $h(\underline{y}, b^*) > b^*$. The lender assesses this probability at time 0, and discounts the debt accordingly, i.e for any $b > 0$ the debt receives a discount reflecting this potential default risk.

$\square$

Proposition 2 (Zero Extreme Value Shock)**.** *As $\alpha \to 0$, the decision problem converges to the one presented in Section 2.*

Proof of Proposition 2. Immediate from Iskhakov et al. (2017, Theorem 5). $\square$

Proposition 3 (Monotonicity)**.** *In the above environment with $T < \infty$,*

1. **Values and Decision Rules across States** *For all t,*

   1.1 *$V_t^R(y, b)$ is strictly increasing in y and strictly decreasing in b.*

   1.2 *$W_t(b)$ is non-increasing in b.*

   1.3 *$\chi_t(b)$ is non-increasing in b.*

   1.4 *If $q_t(b')$ is non-increasing in b', $h_t(y, b)$ is non-increasing in y and non-decreasing in b.*

2. **Debt Prices across Sates:** *For all t, $q_t(b)$ is non-increasing in b.*

3. **Values and Default over Time:** *If $q_t(b) \geq q_{t+1}(b)$, for all $t < T$ and all (y, b),*

   3.1 *$S_t(y, b) \geq S_{t+1}(y, b)$.*

   3.2 *$Z_t(b) \geq Z_{t+1}(b)$.*

   3.3 *$\chi_t(b) \geq \chi_{t+1}(b)$.*

4. **Debt Prices over Time:** *For all $t < T$, $q_t(b) \geq q_{t+1}(b)$.*

Proof of Proposition 3.1.1. Consider the budget set given by:

$$B_t(y,b) = \left\{(c,b') : 0 \le c \le y - b + q_t(b')[b' - (1-\lambda)b]\right\}$$

Clearly, if $\hat{y} > y$ and $\tilde{b} > b$, then $B_t(y,b) \subseteq B_t(\hat{y},b)$ and $B_t(y,\tilde{b}) \subseteq B_t(y,b)$. This implies the optimal choice $h = h(y,b) \in B_t(\hat{y},b)$, and $\tilde{h} = h(y,\tilde{b}) \in B_t(y,b)$. This implies monotonicity in y since,

$$\begin{aligned}
V_t^R(\hat{y},b) &= u\big(\hat{y} - b + q(\hat{h})[\hat{h} - (1-\lambda)b]\big) + \beta W_{t+1}(\hat{h}) \\
&\ge u\big(\hat{y} - b + q(h)[h - (1-\lambda)b]\big) + \beta W_{t+1}(h) \\
&> u\big(y - b + q(h)[h - (1-\lambda)b]\big) + \beta W_{t+1}(h) \\
&= V_t^R(y,b),
\end{aligned}$$

where the first inequality follows from the definition of optimality, and the second from the strict monotonicity of $u(c)$. This also implies monotonicity in b since,

$$\begin{aligned}
V_t^R(y,b) &= u\big(y - b + q(h)[h - (1-\lambda)b]\big) + \beta W_{t+1}(h) \\
&\ge u\big(y - b + q(\tilde{h})[\tilde{h} - (1-\lambda)b]\big) + \beta W_{t+1}(\tilde{h}) \\
&> u\big(y - \tilde{b} + q(\tilde{h})[\tilde{h} - (1-\lambda)\tilde{b}]\big) + \beta W_{t+1}(\tilde{h}) \\
&= V_t^R(y,\tilde{b}),
\end{aligned}$$

where the first inequality follows from the definition of optimality, and the second from the strict monotonicity of $u(c)$. $\square$

Proof of Proposition 3.1.2. Consider the definition of $W_t(b)$.

$$\begin{aligned}
W_t(b) &= \int\int \max\left\{V_t^R(y,b) + \epsilon^R, V_t^A(y) + \epsilon^A\right\} dF_\epsilon(\epsilon) dF_y(y), \\
&= \int \alpha \ln\left(e^{V^R(y,b)/\alpha} + e^{V^A(y)/\alpha}\right) dF_y(y) - \alpha ln(2).
\end{aligned}$$

The proof is immediate from Proposition 3.1.1. $\square$

Proof of Proposition 3.1.3. Recall the definition of the ex-ante repayment probability,

$$\chi_t(b) = \int \phi_t(y,b) dF_y(y) = \int \frac{e^{V_t^R(y,b)/\alpha}}{e^{V_t^R(y,b)/\alpha} + e^{V_t^A(y)/\alpha}} dF_y(y).$$

The claim follows immediately from Proposition 3.1.1. $\square$

Proof of Proposition 3.1.4. Similar to Chatterjee and Eyigungor (2012), define the loss function of choosing b'_0 instead of b'_1 in state (y, b) by

$$\Delta(b'_0, b'_1|b) = (1-\lambda)b_0\big[q(b'_1) - q(b'_0)\big] + q(b'_0)b'_0 - q(b'_1)b'_1.$$

Observe that Δ does not depend on y given that y is i.i.d. distributed.

For the first part of the proposition, fix y and consider two debt levels $b_0 > 0$ and $b_1 > 0$ such that $b_1 > b_0 > 0$. Assume that in b_0, the borrower optimally chooses b'_0 and obtains a consumption level c_0. This means that for a $\hat{b}' < b'_0$ leading to a consumption level $\hat{c}$, we have by optimality that

$$u(c_0) + \beta W(b'_0) \geq u(\hat{c}) + \beta W(\hat{b}'). \tag{20}$$

As $W(b'_0) \leq W(\hat{b}')$ from Proposition 3.1.2, it must be that $c_0 \geq \hat{c}$. Now observe that $\Delta(b'_0, \hat{b}'|b_0) = c_0 - \hat{c} \geq 0$ and

$$\Delta(b'_0, \hat{b}'|b_0) - \Delta(b'_0, \hat{b}'|b_1) = (1-\lambda)(b_0 - b_1)\big[q(\hat{b}') - q(b'_0)\big] \leq 0,$$

where the inequality comes form the fact that $b_1 > b_0 > 0$ and that $q(b')$ is non decreasing in b'. This means that the loss in b_1 is at least as large as the loss in b_0. With this define $\tilde{c}$ being the consumption level in state b_1 choosing b'_0. By the budget constraint, it directly follows that $\tilde{c} < c_0$ given that $b_1 > b_0 > 0$. Combining this with the strict concavity of $u(\cdot)$, we get

$$\begin{aligned}
u(\tilde{c}) - u(\tilde{c} - \Delta(b'_0, \hat{b}'|b_1)) &> u(c_0) - u(c_0 - \Delta(b'_0, \hat{b}'|b_1)) \\
&\geq u(c_0) - u(c_0 - \Delta(b'_0, \hat{b}'|b_0)) \\
&= u(c_0) - u(\hat{c}) \geq 0,
\end{aligned}$$

where the first inequality comes from $c_0 > \tilde{c}$, the second from $\Delta(b'_0, \hat{b}'|b_1) \geq \Delta(b'_0, \hat{b}'|b_0)$ and the third from the definition of $\Delta(b'_0, \hat{b}'|b_0)$. This means that the wedge in utility between $\tilde{c}$ and $\hat{c} = \tilde{c} - \Delta(b'_0, \hat{b}'|b_1)$ is larger than the wedge in utility between c_0 and $\hat{c} = c_0 - \Delta(b'_0, \hat{b}'|b_0)$. By (20), this implies that

$$u(\tilde{c}) + \beta W(b'_0) > u(\hat{c}) + \beta W(\hat{b}').$$

Hence it cannot be that in b_1, the optimal choice b' is lower than b_0'. The bond policy function is therefore non decreasing in b.

For the second part of the proposition, fix b and consider two income levels $y_0 > 0$ and $y_1 > 0$ such that $0 < y_1 < y_0$. As before, assume that in y_0, the borrower optimally chooses b_0' and obtains a consumption level c_0. Considering a $\hat{b}' < b_0'$ leading to a consumption level $\hat{c}$, we get the same argument around (20) as before. With this define $\tilde{c}$ being the consumption level in state y_1 choosing b_0'. From the budget constraint, $\tilde{c} = c_0 + y_1 - y_0 < c_0$ as $0 < y_1 < y_0$. Combining this with the strict concavity of $u(\cdot)$, we get

$$\begin{aligned} u(\tilde{c}) - u(\tilde{c} - \Delta(b_0', \hat{b}'|b) + y_1 - y_0) &> u(c_0) - u(c_0 - \Delta(b_0', \hat{b}'|b) + y_1 - y_0) \\ &> u(c_0) - u(c_0 - \Delta(b_0', \hat{b}'|b)) \\ &= u(c_0) - u(\hat{c}) \geq 0. \end{aligned}$$

The wedge in utility between $\tilde{c}$ and $\hat{c} = \tilde{c} - \Delta(b_0', \hat{b}'|b) + y_1 - y_0$ is larger than the wedge in utility between c_0 and $\hat{c} = c_0 - \Delta(b_0', \hat{b}'|b)$. By (20), it cannot be that in y_1, the optimal choice b' is lower than b_0'. The bond policy function is therefore non increasing in y. □

Proof of Proposition 3.2. We prove this statement by taking the limit of finite horizon. First observe that for $b \leq 0$, $q_t(b) = \bar{p} \sum_{k=0}^{T-t-1} (\bar{p}(1-\lambda))^k$. This implies that for any $b_1 < b_2 \leq 0$, $q_t(b_1) - q_t(b_2) = 0$. Now consider $b > 0$. The bond price is given by

$$q_t(b) = \overline{p}\ \chi_{t+1}(b) + \overline{p}(1-\lambda) \int_{\underline{y}}^{\overline{y}} \phi_{t+1}(y, b)\ q_{t+1}\left[h_{t+1}(y, b)\right]\ dF_y.$$

The proof goes by backward induction. In the last period T, the borrower does not necessarily default for every $b > 0$. This depends on the exact realization of (ϵ^R, ϵ^A). Nevertheless, bonds issued in $T-1$ are one period, given that T is the last period. Hence, the bond price in $T-1$ is given by

$$q_{T-1}(b) = \overline{p}\ \chi_T(b).$$

From Proposition 3.1.3, for any $0 < b_1 < b_2$,

$$q_{T-1}(b_1) - q_{T-1}(b_2) = \overline{p}\left[\chi_T(b_1) - \chi_T(b_2)\right] \geq 0.$$

Subsequently, in period $T-1$, we have that

$$
\begin{aligned}
q_{T-2}(b_1) - q_{T-2}(b_2) =& \bar{p}\left[\chi_{T-1}(b_1) - \chi_{T-1}(b_2)\right] \\
&+ \bar{p}(1-\lambda)\int_{\underline{y}}^{\bar{y}} \phi_{T-1}(y,b_1)q_{T-1}(h_{T-1}(y,b_1)) - \phi_{T-1}(y,b_2)q_{T-1}(h_{T-1}(y,b_2))dF_y \\
\geq& \bar{p}\left[\chi_{T-1}(b_1) - \chi_{T-1}(b_2)\right] \\
&\underbrace{+ \bar{p}(1-\lambda)\int_{\underline{y}}^{\bar{y}} \phi_{T-1}(y,b_2)\left\{q_{T-1}(h_{T-1}(y,b_1)) - q_{T-1}(h_{T-1}(y,b_2))\right\}dF_y}_{\equiv A},
\end{aligned}
$$

where the inequality comes from Proposition 3.1.3. Relying on the same corollary we have that $\chi_{T-1}(b_1) - \chi_{T-1}(b_2) \geq 0$. In addition, from Proposition 3.1.4, $h_{T-1}(y,b_1) \leq h_{T-1}(y,b_2)$ implying that $A \geq 0$ given that $q_{T-1}(b_1) - q_{T-1}(b_2) \geq 0$ for $0 > b_1 > b_2$. Hence,

$$q_{T-2}(b_1) - q_{T-2}(b_2) \geq 0.$$

Repeating the argument until $t = 0$ completes the proof. □

Proof of Proposition 3.3.1. Suppose that at time t, the borrower chooses a sub-optimal strategy to mimic the optimal policy of the subsequent period c_{t+1}, from time t to $T-1$. This is feasible as $q_t(b) \geq q_{t+1}(b)$. Thus, consumption at time $t, \ldots, T-1$ is the same as the optimal consumption for a borrower in the same state at time $t+1$ to T. This implies the value of consumption from $t+1$ to T of a sovereign entering state (y,b) at time t, which follows this sub-optimal strategy, is equal to the value of an optimizing sovereign from $t+1$ to T entering the same state (y,b) at time $t+1$. What's more, the sovereign at time t also has an additional period of consumption $\tilde{c}_T > 0$. As this strategy is sub-optimal, we have that

$$
\begin{aligned}
V_t^R(y,b) &\geq u(c_{t+1}(y,b)) + \mathbb{E}_t\Big\{\sum_{j=t+1}^{T-1} \beta^{j-t}u(c_{j+1}(y_j,b_j)) + \beta^{T-t}u(\tilde{c}_T(y_T,b_T))\Big\} \\
&= V_{t+1}^R(y,b) + \mathbb{E}_t\beta^{T-t}u(\tilde{c}_T(y_T,b_T)).
\end{aligned}
$$

The expectation term is over both the income and the utility shock. From the previous equation, we obtain that

$$V_t^R(y,b) - V_{t+1}^R(y,b) \geq \mathbb{E}_t\beta^{T-t}u(\tilde{c}_T(y_T,b_T)). \tag{21}$$

In addition, observe that

$$V_t^A(y) = u(y) + \sum_{j=t+1}^{T} \beta^{j-t} \int_{\underline{y}}^{\overline{y}} u(y')dF_y(y'),$$
$$V_{t+1}^A(y) = u(y) + \sum_{j=t+2}^{T} \beta^{j-t-1} \int_{\underline{y}}^{\overline{y}} u(y')dF_y(y').$$

Hence

$$V_t^A(y) - V_{t+1}^A(y) = \beta^{T-t} \int_{\underline{y}}^{\overline{y}} u(y')dF_y(y'). \tag{22}$$

Note that if default is optimal in T in at least one state, then $\mathbb{E}_t\beta^{T-t}u(y_T(y_T, b_T)) \leq \mathbb{E}_t\beta^{T-t}u(c_T(y_T, b_T))$, otherwise $\mathbb{E}_t\beta^{T-t}u(y_T(y_T, b_T)) < \mathbb{E}_t\beta^{T-t}u(c_T(y_T, b_T))$. Hence, combining (21) and (22) implies that

$$V_t^R(y, b) - V_t^A(y) \geq V_{t+1}^R(y, b) - V_{t+1}^A(y),$$

as desired. □

Proof of Proposition 3.3.2. Consider the definition of $Z_t(b)$.

$$\begin{aligned} Z_t(b) &= \int \max\left\{V_t^R(y, b), V_t^A(y)\right\} - V_t^A(y)dF_y(y) \\ &= \int \max\left\{V_t^R(y, b) - V_t^A(y), 0\right\} dF_y(y). \end{aligned}$$

The proof is immediate from Proposition 3.3.1. □

Proof of Proposition 3.3.3. Recall the definition of the repayment probability,

$$\phi_t(y, b) = \frac{e^{V_t^R(y,b)/\alpha}}{e^{V_t^R(y,b)/\alpha} + e^{V_t^A(y)/\alpha}}.$$

Hence, it suffices to show that

$$V_t^R(y, b) - V_t^A(y) \geq V_{t+1}^R(y, b) - V_{t+1}^A(y),$$

which follows from Proposition 3.3.1. Furthermore, recall that $\chi_t(b) = \int_y \phi_t(y, b)\ dF_y$. Hence, the monotonicity of $\phi_t(y, b)$ in t implies the monotonicity of $\chi_t(b)$ in t. □

Proof of Proposition 3.4. First observe that for $b \leq 0$, $q_t(b) = \bar{p}\sum_{k=0}^{T-t-1}(\bar{p}(1-\lambda))^k$ for all $t < T$ implying that

$$q_t(b) - q_{t+1}(b) > 0.$$

Now for $b > 0$, we develop an inductive argument. The induction hypothesis relies on the fact that, in a given state (y, b), the borrower cannot trade new bonds in T when repaying. This is as if $q_T(b) = 0$ for all b. As a result, the borrower cannot be worse off repaying in $T-1$ than repaying in T in a given state (y, b) as $q_{T-1}(b)$ is bounded below by zero.

To show this, consider that $\alpha = 0$ meaning that we are back to the decision problem of Section 2 without taste shocks. In this case, at T, the borrower defaults for any $b > 0$ implying that $q_{T-1}(b) = 0$ for all $b > 0$. In $T-1$, if the borrower chooses $b' > 0$, its value is $u(y-b) + \beta \int u(y')dF_y(y')$ as $q_{T-1}(b') = 0$. A default is strictly preferable here as $u(y) + \beta \int_{y'} u(y')dF_y(y') > u(y-b) + \beta \int_{y'} u(y')dF_y(y')$. Conversely, if the borrower chooses $b' < 0$, its values reads $u(y - b + q_{T-1}[b' - (1-\lambda)b]) + \beta \int_{y'} u(y'-b')dF_y(y')$ where $q_{T-1} = \frac{1}{1+r}$. The borrower might not necessarily default in $T-1$ for sufficiently low b. Hence, $q_{T-2}(b) \geq 0$ for all $b > 0$ leading to

$$q_{T-2}(b) - q_{T-1}(b) \geq 0.$$

When $\alpha \neq 0$, a similar argument applies. The only difference is that $q_{T-1}(b) \geq 0$ for all $b > 0$ meaning that the borrower can borrow (and not only save) in $T-1$. As it remains true that $q_T(b) = 0$ for all b, the borrower cannot be worse off repaying in $T-1$ than repaying in T in a given state (y, b). Hence, $\chi_{T-1}(b) - \chi_T(b) \geq 0$ for all b which implies that

$$\begin{aligned} q_{T-2}(b) - q_{T-1}(b) =& \bar{p}\left[\chi_{T-1}(b) - \chi_T(b)\right] \\ &+ \bar{p}(1-\lambda)\int_{\underline{y}}^{\bar{y}} \phi_{T-1}(y,b) q_{T-1}(h_{T-1}(y,b)) dF_y \geq 0. \end{aligned}$$

The above argument gives us the induction hypothesis. We then simply need to show that if $q_{k+1}(b) \geq q_{k+2}(b)$, for each $k \geq t$, then the bond price satisfies $q_t(b) \geq q_{t+1}(b)$ for every b for all $t < T-2$. For this, observe that

$$\begin{aligned} q_t(b) - q_{t+1}(b) =& \bar{p}\left[\chi_{t+1}(b) - \chi_{t+2}(b)\right] \\ &+ \bar{p}(1-\lambda)\int_{\underline{y}}^{\bar{y}} \phi_{t+1}(y,b) q_{t+1}(h_{t+1}(y,b)) - \phi_{t+2}(y,b) q_{t+2}(h_{t+2}(y,b)) dF_y \end{aligned}$$

$$\geq \underbrace{\bar{p}\left[\chi_{t+1}(b) - \chi_{t+2}(b)\right]}_{\equiv A}$$
$$\underbrace{+\, \bar{p}(1-\lambda) \int_{\underline{y}}^{\bar{y}} \phi_{t+2}(y,b) \left\{ q_{t+1}(h_{t+1}(y,b)) - q_{t+2}(h_{t+2}(y,b)) \right\} dF_y,}_{\equiv B}$$

where the inequality comes from Proposition 3.3.3 using the induction hypothesis. Relying on the same argument, $A \geq 0$. We next show that $B \geq 0$ with the following lemma.

Lemma 2. *If $q_t(b) \geq q_{t+1}(b)$ for each $t < T$ and $b < b^{**}_{t+1}$, then $h_t(y,b)$ is such that $q_t(h_t(y,b)) \geq q_{t+1}(h_{t+1}(y,b))$.*

Proof of Lemma 2. Fix (y,b) and assume by contradiction that $h_t(y,b)$ is such that $q_t(h_t(y,b)) < q_{t+1}(h_{t+1}(y,b))$. This implies by Proposition 3.2. that $h_t(y,b) > h_{t+1}(y,b)$. Denote the consumption at time t when the bond choice is $h_t(y,b)$ by $c_t(h_t(y,b))$. We now consider two cases:

1. $c_t(h_t) \leq c_{t+1}(h_{t+1})$

   The contradiction is immediate. The borrower is strictly better off choosing h_{t+1} in t than h_t. The assumption that $c_t(h_t) \leq c_{t+1}(h_{t+1})$ implies that

   $$q_t(h_t)(h_t - (1-\lambda)b) < q_{t+1}(h_{t+1})(h_{t+1} - (1-\lambda)b).$$

   As $q_t(h_t) < q_{t+1}(h_{t+1})$, $(h_{t+1} - (1-\lambda)b) > 0$ meaning that there is no buyback with h_{t+1}. This further implies that by Proposition 3.2. that

   $$q_{t+1}(h_{t+1})(h_{t+1} - (1-\lambda)b) \leq q_t(h_{t+1})(h_{t+1} - (1-\lambda)b),$$

   which in turn implies that $c_t(h_{t+1}) \geq c_{t+1}(h_{t+1}) \geq c_t(h_t)$. In addition, given that $h_t > h_{t+1}$, Proposition 3.1.2 implies that $W_{t+1}(h_t) \leq W_{t+1}(h_{t+1})$. Hence,

   $$u(c_t(h_{t+1})) + \beta W_{t+1}(h_{t+1}) \geq u(c_t(h_t)) + \beta W_{t+1}(h_t).$$

   We rule out the inequality by assuming that the borrower always chooses the lowest level of debt in case of a tie. Hence, the choice h_t is not optimal, a contradiction.

2. $c_t(h_t) > c_{t+1}(h_{t+1})$

To show that there is a contradiction, we first need the following intermediate result.

Claim 1. *There exists a $\tilde{h}_t < h_t$ wich leads to a consumption level $c_t(\tilde{h}_t) < c_t(h_t)$ such that*

$$u(c_t(h_t)) - u(c_t(\tilde{h}_t)) \leq \beta\big[Z_{t+1}(h_{t+1}) - Z_{t+2}(\tilde{h}_t)\big], \tag{23}$$

and

$$Z_{t+1}(h_{t+1}) - Z_{t+2}(\tilde{h}_t) \leq Z_{t+2}(\tilde{h}_t) - Z_{t+1}(h_t). \tag{24}$$

Proof of Claim 1. Note that neither (23) nor (24) put any restriction on the sign of $\tilde{h}_t - h_{t+1}$ given Proposition 3.1.2 and Proposition 3.3.2. The same holds true for the sign of $c_t(\tilde{h}_t) - c_{t+1}(h_{t+1})$.

Observe that equation (23) is satisfied in the interval $[\tilde{b}, h_t]$ where

$$u(c_t(h_t)) - u(c_t(\tilde{h}_t)) = \beta\big[Z_{t+1}(h_{t+1}) - Z_{t+2}(\tilde{b})\big].$$

Conversely, equation (24) is satisfied in the interval $[\bar{h}, \tilde{\tilde{b}}]$ with $\bar{h} < h_{t+1}$ where

$$Z_{t+1}(h_{t+1}) - Z_{t+2}(\tilde{\tilde{b}}) = Z_{t+2}(\tilde{\tilde{b}}) - Z_{t+1}(h_t).$$

A sufficient condition for such a $\tilde{h}_t$ to exist is to have $Z_{t+2}(\tilde{\tilde{b}}) < Z_{t+2}(\tilde{b})$.

$$\beta Z_{t+2}(\tilde{b}) = \beta Z_{t+1}(h_{t+1}) + u(c_t(\tilde{h}_t)) - u(c_t(h_t)).$$
$$\beta Z_{t+2}(\tilde{\tilde{b}}) = \frac{1}{2}\beta\big[Z_{t+1}(h_t) + Z_{t+1}(h_{t+1})\big].$$

The sufficient condition is therefore satisfied whenever

$$\begin{aligned} u(c_t(\tilde{h}_t)) - u(c_t(h_t)) >& \frac{1}{2}\beta\big[Z_{t+1}(h_t) - Z_{t+1}(h_{t+1})\big] \\ =& \frac{1}{2}\beta\big[W_{t+1}(h_t) - W_{t+1}(h_{t+1})\big] \leq 0, \end{aligned}$$

where the equality comes from the definition of Z_{t+1} and the last inequality comes from the fact that $h_t > h_{t+1}$. However, we rule out the fact that $W_{t+1}(h_t) - W_{t+1}(h_{t+1}) = 0$ given that $q_t(h_t) < q_{t+1}(h_{t+1}) \leq q_t(h_{t+1})$ and $c_t(h_t) > c_t(\tilde{h}_t)$. Thus, there exists a $\tilde{h}_t$ such that both equations (23) and (24) are satisfied. $\square$

Given equation (23), we have

$$u(c_t(h_t)) + \beta Z_{t+2}(\tilde{h}_t) \leq u(c_t(\tilde{h}_t)) + \beta Z_{t+1}(h_{t+1}),$$

Observe that, given equation (24), from Proposition 3.3.2

$$\begin{aligned} Z_{t+1}(h_{t+1}) - Z_{t+2}(\tilde{h}_t) &\leq Z_{t+2}(\tilde{h}_t) - Z_{t+1}(h_t), \\ &\leq Z_{t+1}(\tilde{h}_t) - Z_{t+1}(h_t). \end{aligned}$$

Define ϖ such that $Z_{t+2}(\tilde{h}_t) + \varpi = Z_{t+1}(h_t)$. Hence, $Z_{t+1}(h_{t+1}) + \varpi \leq Z_{t+1}(\tilde{h}_t))$. Thus adding $\beta\varpi$ on both sides of the previous expression leads to

$$u(c_t(h_t)) + \beta Z_{t+1}(h_t) \leq u(c_t(\tilde{h}_t)) + \beta Z_{t+1}(\tilde{h}_t).$$

As $Z_{t+1}(h_t) - Z_{t+1}(\tilde{h}_t) = W_{t+1}(h_t) - W_{t+1}(\tilde{h}_t)$, we have that

$$u(c_t(h_t)) + \beta W_{t+1}(h_t) \leq u(c_t(\tilde{h}_t)) + \beta W_{t+1}(\tilde{h}_t).$$

We rule out the inequality by assuming that the borrower always chooses the lowest level of debt in case of a tie. This contradicts the fact that h_t is optimal.

$\square$

Thus, $B \geq 0$ and the bond price q_t is monotonic in t. $\square$

Proposition 4 (Existence and Uniqueness)**.** *Let q^* be the limiting price function of the finite horizon problem, and $\Omega = [\underline{y}, \overline{y}] \times [\underline{b}, \overline{b}]$ be the state space. For a function $S : \Omega \to \mathbb{R}$ defined as $S(y, b) = V^R(y, b) - V^A(y)$, define the operator $(KS)(\Omega; q^*)$ as:*

$$(KS)(y, b; q^*) = \sup_{b'} \left\{ u(y - b + q^*(b')[b' - (1 - \lambda)b]) - u(y) + \beta \int_{\underline{y}}^{\overline{y}} \max\left\{ S\left(y', b'; q^*\right), 0 \right\} dF_y\left(y'\right) \right\}.$$

Then (i) the limit of the finite horizon problem is the fixed point of K given by $\overline{S}(y, b; q^*)$, and (ii) this fixed point exists and is unique.

Proof of Proposition 4. We prove this proposition in three parts. We first establish that the uniform convergence of the bond price. Second, we show that the operator K is c contraction for a given q. Finally, we show that this operator converges when the bond price converges.

Theorem 1 (Dini)**.** *Suppose the sequence of continuous functions $f_n : \mathbb{K} \to \mathbb{R}$ converges pointwise to a continuous function $f : \mathbb{K} \to R$ where $\mathbb{K}$ is a compact set, and furthermore $f_{n+1}(x) \geq f_n(x)$ for every $x \in \mathbb{K}$, then f_n converges uniformly to f.*

Corollary 1 (Uniform Convergence of $q_t(b)$ in t)**.** *The price $q_t(b)$ converges uniformly to a limiting price q^* as the horizon of the problem goes to infinity.*

> *Proof of Corollary 1.* By Proposition 3.4., $q_{n+1}(b) \geq q_n(b)$, where $n = T - t$, as (q_n) is a sequence of nonincreasing and bounded functions it converges pointwise to q^*, and by Dini's Theorem we can strengthen this to uniform convergence. □

Having shown the uniform convergence of the bond price, we now show that the operator K is a contraction given q.

Claim 2. *The operator K is a contraction for a given price q.*

> *Proof of Claim 2.* Recall the definition of the operator K:
>
> $$K(S)(y, b; q) = \max_{c, b' \in \Gamma(y,b;q)} \left\{ u(c) - u(y) + \beta \int \max \left\{ S(y', b'), 0 \right\} dF_y \right\}$$
>
> (i) Monotonicity. Suppose $S_0 \leq S_1$, then:
>
> $$\begin{aligned} K(S_0)(y, b; q) &= \max_{c, b' \in \Gamma(y,b;q)} \left\{ u(c) - u(y) + \beta \int \max \left\{ S_0(y', b'), 0 \right\} dF_y \right\} \\ &\leq \max_{c, b' \in \Gamma(y,b;q)} \left\{ u(c) - u(y) + \beta \int \max \left\{ S_1(y', b'), 0 \right\} dF_y \right\} \\ &= K(S_1)(y, b; q). \end{aligned}$$
>
> (ii) Discounting. Let $a > 0$, then:
>
> $$\begin{aligned} K(S + a)(y, b; q) &= \max_{c, b' \in \Gamma(y,b;q)} \left\{ u(c) - u(y) + \beta \int \max \left\{ S(y', b') + a, 0 \right\} dF_y \right\} \\ &\leq \max_{c, b' \in \Gamma(y,b;q)} \left\{ u(c) - u(y) + \beta \int \left[\max \left\{ S(y', b'), 0 \right\} + a \right] dF_y \right\} \end{aligned}$$

$$
\begin{aligned}
&= \max_{c,b' \in \Gamma(y,b;q)} \left\{ u(c) - u(y) + \beta \int \max\left\{ S(y', b'), 0 \right\} dF_y + \beta a \right\} \\
&= K(S)(y, b; q) + \beta a.
\end{aligned}
$$

□

We finally establish the convergence property of the operator K.

Claim 3. *The operator K is continuous in q.*

Proof of Claim 3. The idea here is similar to Chatterjee and Eyigungor (2012), we fix a choice b' and consider $S_{0,b'}(y, b, ; q^n)$, that is, the value of picking b' today and following the program in all future periods. Clearly, $S_{0,b'}$ is continuous in q^n because the choice set for consumption c is continuous in q^n and utility $u(c)$ is continuous in c. Next we consider $K(S_0)(y, b; q^n) = \sup_{b'} S_{0,b'}(y, b, ; q^n)$, which by the Theorem of the Maximum is continuous in q^n. □

Corollary 2. *If $q^n \to q^*$ and $\bar{S}(y, b; q^n)$ is a fixed point of K, then $\bar{S}(y, b; q^n) \to \bar{S}(y, b; q^*)$.*

Proof of Corollary 2. Direct from the fact that the operator is continuous in q. See Theorem 4.3.6 in Hutson and Pym (1980). □

Taken together with Corollary 1, Claims 2 and 3, and Corollary 2 show there is (i) a unique sequence of prices $\{q_n\}$ with limit q^* implied by the finite horizon problem, and (ii) the value of the sovereign has a unique limiting fixed point $\bar{S}(y, b; q^*)$.

□

Proposition 5 (Convexity)**.** *If α is sufficiently large, then the decision problem is convex.*

Proof of Proposition 5. Observe that

$$
\lim_{\alpha\to\infty} \frac{G(y,b)}{\alpha} = \lim_{\alpha\to\infty} \frac{1}{\alpha} \int_{\epsilon} V(y, b, \epsilon)\, dF_{\epsilon} = \lim_{\alpha\to\infty} \ln\left(e^{V^R(y,b)/\alpha} + e^{V^A(y)/\alpha} \right) - \ln(2) = \ln(2) - \ln(2) = 0.
$$

As noted by Iskhakov et al. (2017), this implies that $V^R(y, b)$ directly inherits the global concavity from the properties of the utility function when the scale parameter is sufficiently large.

In addition, with $\alpha \to \infty$, the borrower decides to default with probability 0.5 irrespective of (y, b). The bond price for any $b' > 0$ is therefore $q(b') \equiv q = \frac{1}{1+2r}$ and the budget set is convex. □

Proposition 6 (Smoothness)**.** *If $b \geq 0$, ϕ, h and q are of class $\mathcal{C}^\infty$ almost everywhere.*

Proof of Proposition 6. The proof goes by backward induction.

- At time T:

  There is no continuation value as the world ends. As a result, $h_T(y, b) = 0$ for all $y \in Y$ and all $b \in B$. The bond policy function $h_T(y, b)$ is trivially $\mathcal{C}^\infty$. However, the sovereign does not necessarily default for all $b > 0$. This depends on the realization of (ϵ^R, ϵ^A). The repayment probability is given by

  $$\phi_T(y, b) = \frac{e^{u(y-b)/\alpha}}{e^{u(y-b)/\alpha} + e^{u(y)/\alpha}}.$$

  We can derive an expression for its first derivative with respect to b

  $$\begin{aligned}\phi_{b,T}(y, b) &= -\frac{u_c(c_T)}{\alpha}\frac{e^{u(y-b)/\alpha}}{e^{u(y-b)/\alpha} + e^{u(y)/\alpha}}\left[1 - \frac{e^{u(y-b)/\alpha}}{e^{u(y-b)/\alpha} + e^{u(y)/\alpha}}\right] \\ &= -\frac{u_c(c_T)}{\alpha}\phi_T(y, b)[1 - \phi_T(y, b)].\end{aligned}$$

  The differentiability of $\phi_T(y, b)$ directly depends on the differentiability of u. Given that u is $\mathcal{C}^\infty$, $\phi_T(y, b)$ is of class $\mathcal{C}^\infty$, except for a jump at $b = 0$. Given this, the bond price at time $T - 1$ corresponds to the price of a one-period bond

  $$q_{T-1}(b) = \begin{cases} \overline{p}\chi_T(b) & \text{if } b > 0 \\ \overline{p} & \text{else} \end{cases}$$

  Except for a jump discontinuity at $b = 0$, $q_{T-1}(b)$ is of class $\mathcal{C}^\infty$ given the property of $\phi_T(y, b)$. As we will see, the former discontinuity is the reason why this proposition only applies to $b \geq 0$.

- At time $T - 1$:

  The optimal borrowing policy is determined by the GEE. In points such that $b_T < 0$, the optimal borrowing is the solution to (EE),

  $$\begin{aligned}&g^{EE}_{T-1}(b_T | b_{T-1}, y) = \\ &u_c(c_{T-1})\overline{p}(1 + (1 - \lambda)\overline{p}) - \beta \int_{\underline{y}}^{\overline{y}} u_c(c_T)[1 + (1 - \lambda)q_T(h_T(y', b_T))]dF_y = 0.\end{aligned}$$

We observe that the integral on the right-hand side contains the price q_T and the bond policy function h_T. The price admits a jump discontinuity at $b = 0$ which is an issue as the integral will not be differentiable at this point. That is why this proposition restricts its attention to $b \geq 0$.

Conversely, in points such that $b_T > 0$, the optimal borrowing is the solution to (GEE),

$$
\begin{aligned}
& g^{GEE}_{T-1}(b_T|b_{T-1}, y) = \\
& u_c(c_{T-1})\Bigg[q_{T-1}(b_T) + \Big\{\overline{p}(1-\lambda)\int_{y'} \phi_T(y', b_T) q_{b,T}(h_T(y', b_T)) h_{b,T}(y', b_T) dF_y \\
& + \overline{p}\int_{y'} \phi_{b,T}(y', b_T)[1 + (1-\lambda) q_T(h_T(y', b_T))] dF_y\Big\}(b_T - (1-\lambda) b_{T-1})\Bigg] \\
& - \beta \int_{y'} \phi_T(y', b_T) u_c(c_T)[1 + (1-\lambda) q_T(h_T(y', b_T))] dF_y = 0.
\end{aligned}
$$

The existence of the derivative $q_{b,T}$ and $h_{b,T}$ is ensured by the fact that q_T and h_T are both of class $\mathcal{C}^\infty$ almost everywhere. The price admits a jump discontinuity which is not an issue as it is outside the interval considered given that $b \geq 0$. Moreover, the repayment probability ϕ_T is $\mathcal{C}^\infty$ except for a kink at 0 which is outside the interval considered. Moreover, the derivative of g^{GEE}_{T-1} with respect to b_T is invertible.[13] As a result, $g^{GEE}_{T-1}(b_T|b_{T-1}, y)$ is of class $\mathcal{C}^\infty$ and so does h_{T-1} except for a discontinuity at 0.

The repayment probability is given by

$$
\phi_{T-1}(y, b) = \frac{e^{V^R_{T-1}(y,b)/\alpha}}{e^{V^R_{T-1}(y,b)/\alpha} + e^{V^A_{T-1}(y)/\alpha}}.
$$

As before, we can derive an expression for its first derivative with respect to b

$$
\begin{aligned}
\phi_{b,T-1}(y, b) = & -\frac{u_c(c_{T-1})}{\alpha}[1 + (1-\lambda) q_{T-1}(h_{T-1})] \\
& \frac{e^{V^R_{T-1}(y,b)/\alpha}}{e^{V^R_{T-1}(y,b)/\alpha} + e^{V^A_{T-1}(y)/\alpha}} \left[1 - \frac{e^{V^R_{T-1}(y,b)/\alpha}}{e^{V^R_{T-1}(y,b)/\alpha} + e^{V^A_{T-1}(y)/\alpha}}\right] \\
= & -\frac{u_c(c_{T-1})}{\alpha}[1 + (1-\lambda) q_{T-1}(h_{T-1})]\phi_{T-1}(y, b)[1 - \phi_{T-1}(y, b)].
\end{aligned}
$$

Given that u is $\mathcal{C}^\infty$, $\phi_{T-1}(y, b)$ is of class $\mathcal{C}^\infty$, except for a jump at $b = 0$. Given this repayment

[13] We do not derive the expression for the derivative. The result is immediate after taking the derivative and the fact that the utility function is bijective. The same argument applies to all g functions considered in this section.

probability, the bond price at time $T-2$ reads

$$q_{T-2}(b) = \begin{cases} \overline{p}\chi_{T-1}(b) + \overline{p}(1-\lambda)\int_{y'} \phi_{T-1}(y', b) q_{T-1}(h_{T-1}(y', b)) dF_y & \text{if } b > 0 \\ \overline{p}(1 + (1-\lambda)\overline{p}) & \text{else} \end{cases}$$

When $b > 0$, the price q_{T-1} and the bond policy function h_{T-1} appear in the integral and admit a jump at 0. The jump discontinuity is not an issue as by assumption, $b \geq 0$, meaning that the jump is outside the interval considered. Thus, except for the discontinuity at $b = 0$, $q_{T-1}(b)$ is of class $\mathcal{C}^\infty$.

- At time $T-n$ for $T \geq n \geq 2$:

  We develop an inductive argument for the remaining periods. We want to show that properties of the price function, the repayment probability and the borrowing policy are preserved from one iteration to the other for $T-n$ for all $T \geq n \geq 2$. We have previously shown that properties are preserved throughout T to $T-1$ (i.e. induction hypothesis).

  We have that q_{T-n} and d_{T-n+1} are of class $\mathcal{C}^\infty$ except for discontinuities at 0. Similarly, q_{T-n+1} and h_{T-n+1} are of class $\mathcal{C}^\infty$ except for discontinuities at 0.

  As for every iteration, the optimal borrowing policy is determined by the GEE. In points such that $b_{T-n+1} < 0$, the optimal borrowing is the solution to (EE),

$$g^{EE}_{T-n}(b_{T-n+1}|b_{T-n}, y) =$$
$$u_c(c_{T-n}) q_{T-n}(y, b_{T-n+1}) - \beta \int_{\underline{y}}^{\overline{y}} u_c(c_{T-n+1})[1 + (1-\lambda) q_{T-n+1}(h_{T-n+1}(y', b_{T-n+1}))] dF_y = 0.$$

  We observe that the integral on the right-hand side contains the price q_{T-n+1} and the bond policy function h_{T-n+1}. Those two objects admit discontinuities. Again, as the discontinuity at 0 is a jump for the price, the integral is not differentiable at this point. However, this is out of the scope of this proposition given that we consider $b \geq 0$.

  Moreover, in points such that $b_{T-n+1} > 0$, the optimal borrowing is the solution to (GEE),

$$g^{GEE}_{T-n}(b_{T-n+1}|b_{T-n}, y) = u_c(c_{T-n}) \Bigg[q_{T-n}(b_{T-n+1}) + (b_{T-n+1} - (1-\lambda) b_{T-n})$$

$$\Bigg\{\overline{p}(1-\lambda)\int_{y'}\phi_{T-n+1}(y',b_{T-n+1})q_{b,T-n+1}(h_{T-n+1}(y',b_{T-n+1}))h_{b,T-n+1}(y',b_{T-n+1})dF_y$$

$$+\overline{p}\int_{y'}\phi_{b,T-n+1}(y',b_{T-n+1})[1+(1-\lambda)q_{T-n+1}(h_{T-n+1}(y',b_{T-n+1}))]dF_y\Bigg\}\Bigg]-\beta$$

$$\int_{y'}\phi_{T-n+1}(y',b_{T-n+1})u_c(c_{T-n+1})[1+(1-\lambda)q_{T-n+1}(h_{T-n+1}(y',b_{T-n+1}))]dF_y=0.$$

The existence of the derivative $q_{b,T-n+1}$ and $h_{b,T-n+1}$ is ensured by the fact that q_{T-n+1} and h_{T-n+1} are both of class $\mathcal{C}^\infty$ except for discontinuities at 0. Given that $b \geq 0$ by assumption, the discontinuity at 0 is not an issue. Moreover, the derivative of g^{GEE1}_{T-n} with respect to b_{T-n} is invertible. As a result, $g^{GEE2}_{T-n}(b_{T-n+1}|b_{T-n},y)$ is of class $\mathcal{C}^\infty$ and so does h_{T-n} by application of the IFT. Thus, $h_{T-n}(y,b)$ is of class $\mathcal{C}^\infty$ and admits a discontinuity at 0.

The repayment probability admits one discontinuity at 0 and is given by

$$\phi_{T-n}(y,b)=\frac{e^{V^R_{T-n}(y,b)/\alpha}}{e^{V^R_{T-n}(y,b)/\alpha}+e^{V^A_{T-n}(y)/\alpha}}.$$

As before, given that u is $\mathcal{C}^\infty$, $\phi_{T-n}(y,b)$ is of class $\mathcal{C}^\infty$, except for a jump at $b=0$. Given this repayment probability, the bond price at time $T-n-1$ reads

$$q_{T-n-1}(b)=q\begin{cases}\overline{p}\chi_{T-n}(b)+\overline{p}(1-\lambda)\int_{y'}\phi_{T-n}(y',b)q_{T-n}(h_{T-n}(y',b))dF_y & \text{if } b>0\\ \overline{p}\sum_{i=0}^{T-(T-n)}[(1-\lambda)\overline{p}]^i & \text{else}\end{cases}$$

The price q_{T-n} and the bond policy function h_{T-n} appear in the integral. Those two objects are of class $\mathcal{C}^\infty$ but admit a discontinuity at 0 which is outside the interval considered. Thus, except for a discontinuity at $b=0$, $q_{T-n}(b)$ is of class $\mathcal{C}^\infty$.

We therefore conclude that the properties of the price function, the repayment probability and the borrowing policy are preserved from one iteration to the other for $T-n$ with any $T \geq n \geq 2$.

□

Proposition 7 (Limiting Bond Policy)**.** *The bond policy function, h_t, converges pointwise to a function h and the derivative of the bond policy function, $h_{b,t}$, converges pointwise to $h_b=\frac{d}{db}h$.*

Proof of Proposition 7. We prove this statement in two parts. First, we show that the implicit function theorem implies some convergence properties (i.e. Lemma 3). Second, we show that as the GEE converges pointwise, so does the bond policy function.

Lemma 3. *Consider a sequence of functions $k_n(x)$ where each k_n is defined by the Implicit Function Theorem from a particular C^r function $f_n(x,y)$, that is, $f_n(x,k_n(x))=0$. If f_n converges (pointwise) uniformly, then k_n converges (pointwise) uniformly.*

Proof of Lemma 3. Suppose that f_n converges uniformly to a function f, i.e. $\lim_{n\to\infty} f_n = f$. By application of the Implicit Function Theorem, there exists a unique k such that $f(x,k(x))=0$. Assume by contradiction that k_n does not converge uniformly. This means that there exists an x such that

$$\lim_{n\to\infty} \sup\{|k_n(x)-k(x)|\} \neq 0.$$

This implies that

$$\lim_{n\to\infty} \sup\{|f_n(x,k_n(x))-f(x,k(x))|\} \neq 0.$$

This directly contradicts the fact that k_n is defined by the Implicit Function Theorem from $f_n(x,y)$ as the implicit function is uniquely defined. Thus, if f_n converges uniformly, so does k_n. Pointwise convergence directly follows from uniform convergence.

$\square$

The GEE is defined over the space (y,b,b') and we know that $GEE(h(y,b)|y,b)=0$ with $b'=h(y,b)$. Hence, for all (y,b) in the point a such that $GEE(a|y,b)=0$, the GEE converges pointwise to zero. Hence, the bond policy function, h_n, converges pointwise by application of Lemma 3.

Now assume by contradiction that $h'_n=\frac{d}{db}h_n$ does not converge to $g=\frac{d}{db}h$ where $h=\lim_{n\to\infty} h_n$. From the Implicit Function Theorem, we know that h is unique. Moreover, as real limits are unique when they exist, g is unique as well. Does this mean we reach a contradiction? Not directly as $\frac{d}{db}\lim_{n\to\infty} h_n(b)$ is not necessarily equal to $\lim_{n\to\infty}\frac{d}{db}h_n(b)$.

To obtain a contradiction observe that the GEE at time t depends on h_{t+1} and h'_{t+1}. Hence, the contradicting assumption leads to

$$\lim_{n\to\infty} GEE_n(h_{n+1},h'_{n+1}) \neq GEE(h,g).$$

This directly contradicts the fact that the GEE converges pointwise to zero given that both h and g are uniquely defined such that $GEE(h,g)=0$. Thus, h_n' converges pointwise to g.

□

Proposition 8 (Limiting Bond Price)**.** *Let $\{q_n\}$ and $\{q_{b,n}\}$ denote the sequence of bond price and its derivative. q_n converges uniformly to q^* and $q_{b,n}$ converges uniformly to q_b^*, implying that $q_b^*=\frac{d}{db}q^*$.*

Proof of Proposition 8. Given Proposition 7, we can elaborate on the limit of the price being differentiable that is:

$$q(b)=\bar{p}\int \phi\left[1+(1-\lambda)q\left(h\right)\right]dF$$
$$q_b=\bar{p}\int \phi_b\left[1+(1-\lambda)q\left(h\right)\right]dF+\bar{p}\int \phi\left[1+(1-\lambda)q_b\left(h\right)h_b\right]dF.$$

Define the operator $T(Q)(b;h,h_b)$ over the space of pairs of continuous bounded functions equipped with the uniform metric, where $Q=(q,q_b)$ and h,h_b are fixed parameters as:

$$T(Q)(b;h,h_b)=\left(\begin{array}{c}\bar{p}\int \phi\left[1+(1-\lambda)q\left(h\right)\right]dF\\ \bar{p}\int \phi_b\left[1+(1-\lambda)q\left(h\right)\right]dF+\bar{p}\int \phi\left[1+(1-\lambda)q_b\left(h\right)h_b\right]dF\end{array}\right).$$

Assume T is continuous in h,h_b. Clearly T is a contraction with modulus $\bar{p}$. From Proposition 7, for all finite n, $h_b^n=\frac{d}{db}h$ and $h^n\to h$ and $h_b^n\to g$ then there is a fixed point $\bar{Q}=(\bar{q}(b;h,g),\bar{q}_b(b;h,g))$ that satisfies

$$T(\bar{Q})(b;h,g)=\left(\begin{array}{c}\bar{p}\int \phi\left[1+(1-\lambda)\bar{q}\left(h\right)\right]dF\\ \bar{p}\int \phi_b\left[1+(1-\lambda)\bar{q}\left(h\right)\right]dF+\bar{p}\int \phi\left[1+(1-\lambda)\bar{q}_b\left(h\right)g\right]dF\end{array}\right).$$

Theorem 2 (Rudin)**.** *Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a,b]$ and such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a,b]$. If $\{f_n'\}$ converges uniformly on $[a,b]$, then $\{f_n\}$ converges uniformly on $[a,b]$, to a function f, and*

$$f'(x)=\lim_{n\to\infty}f_n'(x),\quad x\in[a,b].$$

From our definition of T, q^n converges uniformly (therefore pointwise) to q and q_b^n converges uniformly to q_b, moreover from our previous assumptions we know, for all finite n, $q_b^n=\frac{d}{db}q^n$. Then by the previous Theorem $q_b=\frac{d}{db}\bar{q}$. □