The Generalized Euler Equation and the Bankruptcy-Sovereign Default Problem*

Xavier Mateos-Planas[†] Sean McCrary[‡] José-Víctor Ríos-Rull[§] Adrien Wicht[¶]

June 1, 2026

Abstract

We characterize the equilibrium of the standard sovereign default model with long-term, non-contingent debt. We show existence of the Markov equilibrium and uniqueness of equilibria that are the limit of finite economies. In general, the price and policy functions exhibit jumps and kinks; a suitable choice of arbitrarily small noise yields price and policy functions that are differentiable everywhere, which allows us to characterize the equilibrium using only the agents' decision rules by means of a set of functional equations. We further describe the equilibrium objects via an Euler equation with derivatives on future actions—a Generalized Euler Equation (GEE) that disentangles the effects of default from those of dilution. The GEE yields computational strategies that search for continuous policy functions. A sufficient scale of the noise ensures concavity and a unique solution of the GEE. Applied to a calibrated model following Chatterjee and Eyigungor (2012), the GEE combined with the endogenous grid method delivers residuals orders of magnitude smaller than standard value function iteration, at roughly an order of magnitude lower computational cost.

Keywords: Long-term debt, Sovereign default, Generalized Euler Equation, Computational methods

JEL Classification: F34, E44, C63, G12

---

[*] We thank attendants of seminars at UCL (2015), Southampton Macro Workshop (2016), SED Meetings (Toulouse 2016), EEA Congress (Geneva 2016), and at the 2022 Workshop on Dynamic Games in Macroeconomics and Public Finance (Stony Brook) and Satyajit Chatterjee, Andrew Clausen and especially, Urban Jermann whose presentation of "Sticky Leverage" inspired this project.

[†] Queen Mary University of London, and Centre for Macroeconomics (CFM)

[‡] Ohio State University

[§] University of Pennsylvania, University College, London, CAERP, CEPR and NBER

[¶] University of Basel

1 Introduction

The use of long-term debt (in the spirit of Hatchondo and Martinez (2009), Arellano and Ramanarayanan (2012) and Chatterjee and Eyigungor (2012)) has become widespread in the literature on sovereign default. Competitive risk-neutral lenders with deep pockets trade non-contingent debt contracts with a risk-averse borrower who can always choose to default. The borrower's income is stochastic, so there is value to accessing credit markets, but it cannot commit to repay in future, which affects the current price of debt and hence the terms of that credit access. With long-term debt, the lack of commitment affects not only the repayment of the debt, but also the amount of subsequent borrowing which dilutes the ongoing value of the debt held.

The sovereign's problem, even in its simplest Markovian form,[1] is a particularly fiendish problem. Consumption and bond holdings policy functions do have discontinuities, kinks and flat areas, thereby making the precise representation of its solution and properties challenging.[2] Our approach is closest in spirit to the noise-smoothing route of Chatterjee and Eyigungor (2012): we add small extreme-value shocks to the default decision, which deliver differentiability of policy and price functions without imposing constraints on prices. The purpose of this paper is to provide a precise characterization of the equilibrium, its existence and uniqueness properties and, in particular, of the elements that determine the trade-offs faced by the borrower both in its original form (without extraneous randomness) as well as in the modern form of extreme value shocks to the values of defaulting and not defaulting. An additional objective is to propose computational strategies that, not only offer gains in efficiency and accuracy, but also provide criteria for verifying the computed objects are indeed an equilibrium.

We first lay down the original decision problem describing how the bond price admits points of non-differentiability which affect the optimal borrowing decision. We establish existence of equilibrium and uniqueness of equilibria that are the limit of finite economies, ruling out the type of trigger-strategy equilibria coded as Markov-like equilibria described by Krusell and Smith (2003); our approach mimics the one of Bhaskar et al. (2012). The equilibrium that arises as this limit is of one of two types—a borrowing equilibrium, in which the sovereign enters the risky borrowing region with positive probability and all debt levels carry a discount for dilution risk, or a saving equilibrium, in which the sovereign never exceeds the risky debt limit and so faces no dilution discount—with the type selected by primitives rather than chosen by the limit; Section 4 develops this point and relates it to the multiplicity result of Aguiar

---

[1] Where we are only looking at the Markov equilibria of games with sequences of sovereigns.

[2] To deal with some of these difficulties Chatterjee and Eyigungor (2012) add i.i.d. truncated Normal endowment shocks to the model, and Aguiar and Amador (2021) resort to technical constraints on prices in addition to continuous time and a shock to the outside value of default.

and Amador (2020). We also discuss the implied discontinuities for the policy functions, including kinks and jumps that make the original problem hard to characterize and to solve for (Chatterjee and Eyigungor (2012); Dvorkin et al. (2021)); analytical progress calls for introducing some form of smoothness.

We proceed by posing extreme value shocks (or in general any continuous shocks) to the default decision, thereby transforming the environment to allow for randomization between repayment and default. This change to the environment has many advantages: the choice is always interior, extreme value shocks are well known in the profession and they yield simple formulae for the probabilities of each action. In this transformed economy existence and uniqueness are also established.

We move on to characterize the optimality conditions analytically by means of an intertemporal first-order condition with derivatives on future actions —i.e. a Generalized Euler Equation (GEE). We proceed by first assuming convexity of the decision problem and differentiability of the price, the default set and the bond policy function to derive the GEE, and then we provide conditions under which these two assumptions hold true. The GEE enables a complete characterization of interior solutions and determines the main forces at play. We show that the introduction of extreme value shocks can convexify the decision problem and enables the price and the policy functions to become smooth everywhere. Hence, the proper derivation of the GEE is possible for all positive values of debt. Even when the problem is not convex, the GEE remains useful: it may have multiple solutions, but a simple multi-start safeguard—running the root-finder from low and high initializations and comparing values—identifies the global maximizer (see Section 6.3). This characterization is not confined to the simple environment in which we derive it: Appendix B shows that the same GEE, with the appropriate modifications, characterizes the canonical long-term-debt model with persistent income and stochastic re-entry (Appendix B.1), the partial-default model of Arellano et al. (2023) (Appendix B.2), and the case of a positive recovery rate upon default (Appendix B.3).

The GEE serves both as a basis for computation and as a criterion to verify that a candidate solution is indeed an equilibrium. As a basis for computation it allows one to solve for the policy functions using global methods that avoid discretization. We implement two such methods: policy function iteration on the GEE (PI), and the endogenous grid method of Carroll (2006) adapted to the GEE (EGM). At CRRA $\sigma = 2$ the GEE for consumption reduces to a quadratic, which EGM exploits in closed form; PI proceeds without the closed form by solving the GEE pointwise. Both methods approximate policy, value, and price functions with piecewise cubic Hermite polynomials so that the derivatives required by the GEE are available off-grid. This delivers accurate and fast computation that is not generally possible with discretization of the state space (even with extreme-value smoothing).

The starting point of the literature on sovereign debt and default is the study of Eaton and Gersovitz (1981) which has been directly extended by Aguiar and Gopinath (2006) and Arellano (2008).[3] The initial model considers a sovereign borrower which trades one-period bonds with a continuum of competitive lenders. In this environment, Hatchondo and Martinez (2009), Arellano and Ramanarayanan (2012) and Chatterjee and Eyigungor (2012) introduce long-term debt in the form of a geometrically decreasing maturity rate. This approach is computationally friendly as it cuts down on the number of state variables and enhances the quantitative fit to the data. Nevertheless, analytical characterization of the problem becomes more arduous as the bond price depends on future actions.

Our analysis provides an analytical characterization of incomplete markets models with default under long-term bonds. Such characterization already exists for short-term debt. More precisely, Auclert and Rognlie (2016) show existence and uniqueness in Eaton and Gersovitz (1981). The proof relies on the standard argument made by Bulow and Rogoff (1989) that reputation alone cannot sustain debt. Feng and Santos (2021) extend the existence and uniqueness results to a model with capital, and furthermore show existence of an equilibrium with smooth policies with the addition of taxation. Aguiar and Amador (2019) provide a contraction mapping argument, and Aguiar et al. (2019) show existence and uniqueness by looking at the dual problem of the revenue-maximizing lender. Finally, Clausen and Strub (2020) use reverse calculus and nested optimization. Their paper is the closest in spirit to ours as they also use differentiability to characterize equilibrium, albeit of short term debt (and other environments such as adjustment costs and social insurance). In our environment, long-term bonds introduce the possibility to dilute legacy debt which requires the bond policy to be differentiable alongside the bond price and the default set. With short-term debt, differentiability of the bond policy is not needed. Another important difference is that in Clausen and Strub (2020) the lack of smoothing shocks prevent the use of the GEE because of kinks that have to be found. Their results only speak of situations where the first order condition holds as an equality.

Regarding long-term debt, Chatterjee and Eyigungor (2012) show the existence of an equilibrium bond price. Their proof relies on randomization through continuous *i.i.d.* shocks and restricting the choice of debt to a finite set. This allows for the application of Brouwer's Fixed Point theorem. Subsequently, Aguiar and Amador (2020) show that there exist two equilibria in such class of model: a "borrowing" equilibrium where the borrower issues debt until it reaches some debt limit and defaults, and a "saving"

---

[3] There is an associated large literature on household bankruptcy that is based on essentially the same theory and is supported by the U.S. Bankruptcy code that justifies many of the assumptions made (Livshits et al. (2007); Chatterjee et al. (2007)). There is another literature concerned with lack of commitment with long-term debt like we do here but without default (Debortoli et al. (2017); Gomes et al. (2014); Faraglia et al. (2019)).

equilibrium where the borrower reduces the stock of debt until default no longer occurs with positive probability. In contrast, we show uniqueness by focusing on equilibria that are the limit of those of finite economies, while Aguiar and Amador (2020) might not rule out trigger strategy equilibria coded as Markov-like equilibria.

Our characterization relies on the GEE. More precisely, the Euler equation contains derivatives of prices. We therefore relate to the work on time inconsistent policy of notably Krusell et al. (2002, 2010), Krusell and Smith (2003), Klein et al. (2008), Mateos-Planas (2010) and others. Our work relates to the work of Hatchondo and Martinez (2009), Niepelt (2014), Arellano and Ramanarayanan (2012), and Hatchondo et al. (2016) who study the trade-off between the issuance of short-term and long-term debt and to the work of Arellano et al. (2023) who analyze the decision to partially default. An early version of Aguiar and Gopinath (2006) and Hatchondo et al. (2016) include also a discussion of the first order conditions using debt price derivatives. All the aforementioned studies adopt a heuristic approach by assuming that the relevant policy functions and prices are differentiable ignoring the existence of the critical points or thresholds discussed above. In contrast, we show that those objects are indeed differentiable everywhere provided the default decision is affected by utility shocks. The GEE that we obtain provides a precise decomposition with various additional terms incorporating the default and dilution effects that are novel in this problem.

Beyond its analytical content, the GEE also provides a natural computational strategy to both solve the model as well as to analyze any candidate solution obtained. In this context, the finding of Hatchondo et al. (2010) that different solution methods such as splines or Chebyshev polynomials lead to different numerical results than the standard value function iteration algorithm in the case of short-term debt calls for a criterion under which to assess the extent to which the candidate solutions are indeed equilibria. The GEE provides such a criterion because it is a necessary condition that the solution candidate must satisfy. The strategy followed in Arellano et al. (2016), that use the Euler equation to solve the short-term debt problem numerically assuming that the GEE always holds, can then be verified. We see our contribution as both analytical and computational: the GEE characterization complements and grounds the computational strategies used to solve for equilibrium.

When we solve the model, we rely on extreme value shocks to apply the GEE by means of the two GEE-based methods introduced above — policy function iteration and the endogenous grid method adapted from Carroll (2006). Unlike Arellano et al. (2020), Mihalache (2020) and Dvorkin et al. (2021), we only introduce such shocks in the value of defaulting and repayment. We show that there is a scale of these shocks that is sufficient to ensure strict convexity of the maximization problem. The decision problem might remain non-convex with small scale parameters (because of insufficient utility noise). In this case, the set of solutions to the GEE will not be unique, yet the relevant solution to the GEE (the one associated to the global maximum) tells us how the different elements weight in the optimal choice. We also contribute to this literature by establishing that extreme value shocks enable the differentiability of the bond price and the bond policy function.

Section 2 presents the environment and the decision problem. Section 3 considers the decision problem under extreme value shocks and Section 4 establishes existence and uniqueness as the limit of finite horizon economies. Section 5 derives the GEE and characterizes the equilibrium. Section 6 presents a fully calibrated quantitative application following Chatterjee and Eyigungor (2012), provides detailed pseudocode for the three solution methods compared in the main text (discrete-state value function iteration, policy function iteration, and the endogenous grid method), and compares their accuracy and speed. Section 7 concludes. All the proofs are in the Appendix.

A clarification about the scope of our uniqueness result is in order: it is a selection result showing uniqueness of equilibria that arise as limits of finite-horizon economies, but it does not claim uniqueness of all Markov equilibria in the infinite-horizon model.

2 The Model

We consider a standard model of incomplete markets with default in the spirit of Eaton and Gersovitz (1981). A risk averse sovereign trades long-term bonds with deep pockets risk neutral, competitive lenders. Crucially, the sovereign cannot commit its future selves to repay outstanding debt. We begin by describing the environment and the sovereign's decision problem (Section 2.1), and then define bond prices and the Markov equilibrium (Section 2.2, Definition 1).

For analytical clarity, the model of this section incorporates two simplifying choices. First, the endowment process is i.i.d. Second, default is followed by permanent exclusion from credit markets. Neither restriction is essential for the results developed in Sections 3 and 5. The calibrated quantitative exercise in Section 6 relaxes both restrictions.[4]

---

[4] The first simplification makes the bond price depend only on the level of next-period debt, replacing conditional with unconditional expectations in continuation values. The second simplification makes the autarky value depend only on the current endowment, since exclusion from credit markets is permanent. Together, these two simplifications deliver a closed-form expression for the autarky value as the present discounted value of utility from the endowment alone. The analytical proofs in Section 3–Section 5 go through if the unconditional distribution of the endowment is replaced by its conditional distribution and if the autarky value is replaced by the reentry-augmented expression introduced in Section 6; only the notation becomes heavier, and Appendix B.1 writes out the resulting GEE explicitly. The calibrated quantitative exercise in Section 6 uses a persistent AR(1) endowment process and stochastic reentry to credit markets, together with the standard quadratic output penalty during default.

2.1 Environment and Decisions

There is a continuum of competitive, risk neutral lenders that discount future payoffs at rate $\overline{p} \equiv \frac{1}{1+r}$, where $r = R - 1$ is the exogenous world risk-free interest rate.

The sovereign receives a stochastic endowment $y \in Y \equiv [\underline{y}, \overline{y}] \subset \mathbb{R}^{++}$ which is i.i.d. with *absolutely continuous* distribution function $F^y(y)$ and density $f^y(y) > 0$. Preferences are represented by a utility function $u : \mathbb{R}^{++} \to \mathbb{R}$ that is strongly concave, L-smooth, monotone, of class $\mathcal{C}^\infty$ with strongly convex and L-smooth first derivative. The sovereign discounts the future at rate $\beta < \overline{p}$.[5]

The sovereign trades non-contingent long-term bonds in amounts $b \in B \equiv [0, \overline{b}]$. Following Hatchondo and Martinez (2009), each unit of outstanding debt pays a coupon of one each period and a fraction $\lambda \in [0, 1)$ matures. We restrict to non-negative debt positions, in line with the quantitative sovereign-default literature, where sovereigns are net debtors. The lower bound $b = 0$ is admissible —corresponding to a debt-free initial condition and to the terminal buyback in finite-horizon economies (Section 4)—but on the equilibrium path of the infinite-horizon problem, debt remains in the strictly positive interior (Assumption 1).

To make the value function bounded, we impose loose upper bounds $\overline{b}$ on debt and $\overline{c}$ on feasible consumption, both strictly larger than the present value of the maximum endowment $R\overline{y}/(R-1)$; the value function is then bounded by $\overline{V} = u(\overline{c})/(1-\beta)$. These bounds are technical scaffolding for the boundedness arguments below: they do not restrict equilibrium outcomes because no sovereign can service debt exceeding the present value of its maximum endowment, so equilibrium debt and consumption stay strictly below $\overline{b}$ and $\overline{c}$ respectively. We verify this ex-post in the calibrated quantitative exercises in Section 6.

The sovereign can choose to default on its debt obligations. Imposing a Markovian structure in which only payoff-relevant variables enter the state, the value function is

$$V(y, b) = \max \left\{ V^R(y, b), V^A(y) \right\}, \tag{1}$$

where V^R and V^A denote the values of repayment and default (autarky), respectively. The value of

---

[5] L-smoothness guarantees an upper bound on the second derivative, whereas strong concavity/convexity guarantees a strictly positive lower bound on the absolute value of the second derivative. The vast majority of quantitative studies in the sovereign debt literature uses the CRRA utility $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ with $\sigma > 0$ which satisfies all the listed conditions provided that consumption is bounded.

financial autarky is

$$V^A(y) = u(y) + \beta \, \frac{\int_{\underline{y}}^{\overline{y}} u(y') \, dF^y}{1-\beta},$$

where the second equality follows from the iid structure of the endowment process.

If the sovereign repays, it can issue new long-term debt. The value under repayment is

$$V^R(y,b) = \max_{b' \in \Gamma(y,b;q)} \left\{ u\left(y - b + q(b') \,\left[b' - (1-\lambda)b\right]\right) + \beta \int_{\underline{y}}^{\overline{y}} V(y',b') \, dF^y \right\}, \tag{2}$$

where the feasible set is $\Gamma(y,b;q) = \{b' \in B : 0 \leq y - b + q(b')[b' - (1-\lambda)b] \leq \overline{c}\}$. Here, $q(b')$ is the per-unit price of defaultable debt at total next-period debt b'. Because $b, b' \geq 0$, both issuance and repurchase of debt occur at this same price, and the budget constraint involves no second, risk-free price; an environment that also allowed savings ($b < 0$ or $b' < 0$) would require a distinct price for risk-free transactions and is outside the scope of this paper. The i.i.d. assumption on endowments, together with the restriction to Markovian equilibria, implies that prices depend only on total debt b', not on current income.

The solution to this maximization problem defines the policy function $b' = h(y,b)$. We can then define the default threshold as

$$d(b) = \min \Big\{ \{y : V^R(y,b) \geq V^A(y)\} \cup \{\overline{y}\} \Big\}. \tag{3}$$

That is, for each debt level b, either there exists an income level at which the sovereign is indifferent between repayment and default, or no such point exists, in which case $d(b) = \overline{y}$. Under the i.i.d. endowment assumption, this threshold is independent of current income realisations.

2.2 Equilibrium and Price

Because lenders are risk neutral and competitive, bond prices satisfy a zero expected profit condition. The price of one unit of debt issued at level $b' \geq 0$ is therefore

$$q(b') = \overline{p} \int_{\underline{y}}^{\overline{y}} \mathbb{I}_{\left\{V^R(y',b') > V^A(y')\right\}} \left[1 + (1-\lambda)\, q(b'')\right] dF^y,$$

where $b'' = h(y',b')$, and $\mathbb{I}$ is the indicator function. This expression makes clear that the price depends only on next period's debt b'. Using the default threshold $d(b)$ and the policy function $h(y,b)$, the pricing equations can be written more compactly as

$$q(b') = \overline{p}\left[1 - F^y(d(b'))\right] + \overline{p}\,(1-\lambda)\int_{d(b')}^{\overline{y}} q\left(h(y', b')\right)\, dF^y. \tag{4}$$

Because debt is long-term, debt repayment is spread over multiple periods, and bond prices incorporate the sovereign's future actions. In particular, variation in prices driven by $d(b')$ reflects default risk, while variation driven by $h(y', b')$ captures dilution risk. Dilution refers to the fact that the sovereign cannot commit not to borrow more in the future and, therefore, cannot commit to maintaining a given level of default risk over time.

We can now define a Markov competitive equilibrium.

Definition 1. *A Markov competitive equilibrium in this environment consists of policy functions for the sovereign's debt issuance $h(y, b)$ and default $d(b)$, together with a bond price function $q(b')$, such that*

1. *Taking $q(b')$ as given, the functions $h(y, b)$ and $d(b)$ solve the sovereign's problem in (1)-(3).*
2. *The bond price $q(b')$ satisfies (4), i.e., it correctly reflects default probabilities and yields zero expected profits for lenders.*

2.3 Kinks and Jumps

The bond price exhibits non-smoothness, which gives rise to kinks and discontinuities in the bond policy function. To clarify their origin, it is useful to define two threshold levels. The *default-risk* limit, b^*, is the highest level of debt at which the sovereign does not default in the current period, even at the lowest realization of the endowment. For debt levels below b^*, the sovereign does not default today, although future default remains possible, thereby diluting the value of outstanding debt. To formalise this distinction, define the *dilution-risk* debt limit, b^r, as the highest level of debt such that the sovereign neither defaults today nor at any future date under any sequence of endowment realizations. By construction, $b^r \leq b^*$. The borrowing space can therefore be partitioned as

$$[0, \overline{b}] = [0, b^r] \cup (b^r, b^*] \cup (b^*, \overline{b}],$$

where the first interval is free of both default and dilution risk, the second involves dilution risk but no current default risk, and the third features both default and dilution risk.

The paper focuses on equilibria in which the sovereign enters the default-risk region with positive probability; existence and uniqueness of such an equilibrium are established in Section 4. In such equilibria, dilution risk is priced from the first unit of debt, implying that there is no dilution-risk-free region, i.e., $b^r = 0$.[6] The debt price function $q(b')$ remains smooth at b^r as no discrete change in dilution risk should occur at that point.[7]

In contrast, at the default-risk threshold $b' = b^*$, the derivative $q_b(b')$ exhibits a discrete downward jump. This non-differentiability of the price function has implications for the optimal borrowing choice, generating kinks and discontinuities (or jumps), even if the expected continuation values in (2) and (4) are smooth in b'. The reason is that the sovereign's marginal utility with respect to b' in (2),

$$u_c(c) \times \left[q_b(b')(b' - (1-\lambda)b) + q(b') \right],$$

inherits this discontinuity. At b^*, the discrete drop in $q_b(b')$ induces a corresponding jump in marginal utility. Because the shift in $q_b(b')$ is negative, the resulting change in marginal utility has the opposite sign of $(b' - (1-\lambda)b)$.

At the default-risk limit $b' = b^*$, the implications depend on whether the sovereign is repurchasing or issuing debt. When $b' - (1-\lambda)b < 0$, the sovereign is repurchasing debt. In this case, the discrete decline in $q_b(b')$ at b^* induces a positive jump in marginal utility, which can generate a discontinuous increase in the optimal choice of b'. The magnitude of this jump depends on λ. To build intuition, consider a situation in which the optimal b' approaches b^* from below as b increases. While $b' - (1-\lambda)b < 0$, the sovereign is repurchasing part of its outstanding debt. At the point where the price schedule becomes less favorable discretely, the sovereign has an incentive to sharply reduce repurchases, resulting in a discrete upward jump in b'.

By contrast, when $b' - (1-\lambda)b > 0$, the sovereign is issuing new debt. In this region, the drop in $q_b(b')$ generates a negative discrete change in marginal utility, which can produce a kink and potentially a flat segment at b^* in the policy function. Intuitively, as inherited debt b rises, the sovereign restrains further increases in b' beyond b^* to avoid the discrete deterioration in borrowing terms.

This non-differentiability hampers the characterisation of the equilibrium. As the preceding discussion makes clear, it arises even in the benign case in which the continuation values in the functional equations are smooth.[8] In what follows, we dispense with points of non-differentiability in the bond

---

[6] In the alternative "saving" equilibrium of Aguiar and Amador (2020), in which the sovereign never enters the risky region, $b^r = b^*$ and dilution risk is not priced at any debt level.

[7] If we were to allow for savings, since $b' < 0$ is priced at the risk-free rate $1/(r+\lambda)$, the function $q(b')$ would not only be non-differentiable but also discontinuous at the threshold $b' = b^r (= 0)$.

[8] The situation in general may become worse since the kinks can propagate and render continuation values non smooth (see, e.g., Iskhakov et al. (2017)).

price. In Section 3, we introduce utility shocks that eliminate the threshold b^*.

3 The Model with Extreme Value Shocks

We introduce extreme value taste shocks (hereafter "taste shocks") to the values of repayment and default. Specifically, after the realization of the endowment y, the sovereign receives additive utility shocks $\epsilon = \left(\epsilon^R, \epsilon^A\right)$ to the value of repayment and default, respectively. These shocks are assumed to be i.i.d. Type I Extreme Value (Gumbel). The shocks are normalised so that the expected value of the maximum is zero. This is achieved by setting the location parameter to $-\alpha(\gamma + \ln 2)$, where $\gamma \approx 0.57$ is the Euler-Mascheroni constant and $\alpha > 0$ is the scale parameter.[9] The beginning-of-period value is then

$$V(y, b, \epsilon) = \max\left\{V^R(y, b) + \epsilon^R, V^A(y) + \epsilon^A\right\}.$$

Under this specification, the *ex-post* probability of repayment, $\phi(y, b)$, takes the logit form

$$\phi(y, b) = \frac{e^{V^R(y,b)/\alpha}}{e^{V^R(y,b)/\alpha} + e^{V^A(y)/\alpha}}.$$

This formulation replaces the discrete default decision with a continuous probability. As a result, the default threshold b^* is eliminated, removing the source of non-differentiability in the bond price described in Section 2.3.

Letting F^y and F_ϵ denote the probability measures over endowments and shocks, respectively, define the expected value function

$$G(y, b) \equiv \int_\epsilon V(y, b, \epsilon)\ dF_\epsilon = \alpha \ln\left(e^{V^R(y,b)/\alpha} + e^{V^A(y)/\alpha}\right) - \alpha \ln(2).$$

The additive constant $-\alpha \ln 2$ (together with the Euler–Mascheroni constant absorbed into the location parameter above) does not depend on the borrowing choice, so it drops out of the first-order conditions and hence of the generalized Euler equation and of all the price and value derivatives. To lighten the notation, we therefore suppress it in the inclusive values that appear in the computational formulation of Section 6 and in Appendix B, writing G and the continuation value W as plain log-sum-exp expressions. Moreover, we have[10]

$$\phi(y, b) = G_R(y, b),$$

---

[9] This choice of mean ensures that the option to default does not yield any ex-ante utility or disutility.

[10] This is true for a more general class of taste shocks than Gumbel. See Rust (1988).

where $G_R \equiv \partial G / \partial V^R$ denotes the derivative of the log-sum-exp form of G with respect to V^R. In addition, define the *ex-ante* repayment probability $\chi(b) \equiv \int_y \phi(y, b)\, dF^y$, and the expected continuation value in repayment $W(b) = \int_y G(y, b)\, dF^y$. The value functions under repayment and autarky remain as in Section 2. Finally, given the repayment probability, the bond price for b' is

$$q(b') = \overline{p}\ \chi(b') + \overline{p}(1-\lambda) \int_{\underline{y}}^{\overline{y}} \phi(y', b')\ q\big(h(y', b')\big)\ dF^y. \tag{5}$$

As the scale parameter α decreases towards low values, the model derived in this section converges to the one posed in Section 2.

Proposition 1 (Vanishing Taste Shocks)**.** *As $\alpha \to 0$, the decision problem converges to the one presented in Section 2.*

Moreover, the size of the scale parameter α is critical in ensuring strong concavity of the objective function and convexity of the budget set.[11] When $\alpha \to 0$, the repayment probability becomes degenerate, recovering the discrete default decision studied in Section 2. At the other extreme, as $\alpha \to \infty$, the repayment probability converges to $1/2$ for all states. In this case, the objective function directly inherits the properties of the instantaneous utility function and the budget set becomes convex.

Proposition 2 (Budget Set and Objective Function)**.** *For sufficiently large α, the objective function is strongly concave and L-smooth, the marginal objective is strongly convex in b' and Lipschitz, and the budget set is convex.*

As a consequence, when α is sufficiently large, the bond policy function is uniquely defined and, by Berge's Maximum Theorem, continuous. L-smoothness provides an upper bound on the Hessian, while strong concavity/convexity ensures a strictly positive lower bound on the Hessian—properties that are instrumental in establishing Propositions 4 and 6.

4 The Limit of Finite Horizon Economies

We turn to characterising the sovereign's problem under a finite horizon (Section 4.1) and then study the limit as the horizon tends to infinity (Section 4.2). This will reward us with two results: existence and uniqueness of equilibrium, provided that α is sufficiently large. Given the subtlety of the argument, we proceed in steps, isolating the role of each assumption.

---

[11] See also the discussion in Iskhakov et al. (2017).

4.1 Decision under Finite Horizon

Consider an environment as in Section 3, except that the sovereign and competitive lenders trade long-term debt for a finite number of periods, $T < \infty$. We make two additional assumptions.

First, we focus throughout on equilibria in which the sovereign's debt policy is interior, in the sense that the borrower never chooses to fully de-lever before the terminal date.

Assumption 1 (Interior Debt Policy)**.** *For all (y, b) and all $t < T$, the equilibrium debt policy satisfies $h_t(y, b) > 0$.*

This is an equilibrium-selection device rather than a restriction on primitives. It rules out cases in which the sovereign optimally chooses the corner $b' = 0$ in some state (y, b) at some $t < T$. Two remarks are in order. (i) Under Assumption 1, the Generalized Euler Equation derived in Section 5 characterizes optimality as an equality on $Y \times B$; absent the assumption, the same first-order condition holds as a KKT condition with a non-negative multiplier on the constraint $b' \geq 0$, equal to zero whenever $h_t(y, b) > 0$. (ii) Strict positivity of h_t at the optimum is precisely what allows the implicit function theorem to deliver smoothness of h_t and q_t (Propositions 5 and 6); the assumption is therefore a regularity property of the equilibrium under study, of the same nature as our requirement in Proposition 4 that α be sufficiently large to ensure a unique maximizer. In the calibrated borrowing-type equilibria of Section 6, this property is verified numerically state by state.

Second, in period T, there is no new issuance of bonds, and the borrower repurchases the entire stock of outstanding debt at the equilibrium price consistent with zero future debt.

Assumption 2 (Terminal Period)**.** *In Period T, given state (y, b), consumption is*

$$c_T(y, b) = y - \left[1 + \frac{1-\lambda}{1+2r+\lambda}\right] b.$$

At T, the sovereign repurchases all outstanding debt by choosing $b_{T+1} = 0$ at the price of the marginal unit of debt $q_T(b_{T+1}) = 1/(1 + 2r + \lambda)$ that is consistent with indifference between repaying or not in the future.[12] This assumption preserves the long-term maturity structure for all $t \leq T$. By contrast, if instead $c_T(y, b) = y - b$, then debt issued in $T - 1$ would effectively be one-period debt,

---

[12] The price $\frac{1}{1+2r+\lambda}$ is the unique fixed point of $\bar{q} = \frac{1}{2}\bar{p}[1 + (1-\lambda)\bar{q}]$, corresponding to the long-term bond price when the repayment probability is 1/2 at every future date. This is the self-consistent price at zero debt under extreme value shocks, since $V^R_{T+1}(y, 0) = V^A_{T+1}(y)$ implies $\phi_{T+1}(y, 0) = 1/2$. In contrast, the dilution risk-free price $\frac{1}{r+\lambda}$ would assume sure repayment, which is inconsistent with the taste shock model. This choice also ensures that the bond price is continuous from the right at $b' = 0$, eliminating the discontinuity that would arise under $\frac{1}{r+\lambda}$.

implying that the maturity would vary mechanically with the distance to the terminal date T. Given this, in period T, the beginning-of-period value is

$$V_T(y,b,\epsilon) = \max\left\{u(c_T(y,b)) + \epsilon^R, u(y) + \epsilon^A\right\}.$$

The borrower does not necessarily default on all its debt; it depends on the realisation of $\left(\epsilon^R, \epsilon^A\right)$. In period $T-1$, the value of repayment is

$$V^R_{T-1}(y,b) = \max_{b' \in \Gamma(y,b;q_t)} \left\{ u\left(y - b + q_{T-1}(b')\ \left[b' - (1-\lambda)b\right]\right) + \beta \int_{\underline{y}}^{\overline{y}} \int_{\epsilon} V_T(y',b',\epsilon')\, dF_\epsilon(\epsilon')\, dF^y(y') \right\},$$

and the value of autarky is

$$V^A_{T-1}(y) = u(y) + \beta \int_{\underline{y}}^{\overline{y}} V^A_T(y') dF^y(y'),$$

where $V^A_T(y') = u(y')$ and $\int_\epsilon V_T(y,b,\epsilon')dF_\epsilon(\epsilon') = \alpha \ln(e^{V^R_T(y,b)} + e^{V^A_T(y)}) - \alpha \ln(2)$. These values extend recursively for all $t < T-1$. For $t < T$, the bond price satisfies

$$q_t(b') = \bar{p} \int_{\underline{y}}^{\bar{y}} \phi_{t+1}(y',b')[1 + (1-\lambda)q_{t+1}(h_{t+1}(y',b'))]dF^y(y'), \quad b' \in B.$$

The time subscript reflects the dependence of the repayment probability on t. Under Assumption 2, however, the maturity structure is invariant across time. In period $T-1$, the bond price satisfies

$$q_{T-1}(b') = \bar{p} \int_{\underline{y}}^{\bar{y}} \phi_T(y',b')[1 + (1-\lambda)\frac{1}{1+2r+\lambda}]dF^y(y') = \chi_T(b')\frac{2}{1+2r+\lambda},$$

where $\chi_T(b') \equiv \int_{\underline{y}}^{\bar{y}} \phi_T(y',b')dF^y(y')$. Because the borrower repurchases all outstanding debt at the zero-debt equilibrium price $\frac{1}{1+2r+\lambda}$ and issues no new debt in period T when not defaulting, the continuation price is pinned down by the terminal condition. Finally, as $b' \to 0$, we have $\chi_T(b') \to 1/2$, implying $q_{T-1}(0) = \frac{1}{1+2r+\lambda}$, confirming that the bond price is continuous at zero.

By comparing bond prices, policy functions and value functions across periods, we identify properties linking objects at generic dates t and $t+1$. These properties will allow us to establish that the sequences of policies, value functions, and prices converge to a unique limit. The next section develops these results.

4.2 Monotonicity and Limit of Finite Horizon

To establish existence and uniqueness of the limit of the decision problem as $T \to \infty$, we first show that the main value functions are monotone over y, b and t. To this end, define the repayment surplus as

$$S_t(y, b) = V_t^R(y, b) - V_t^A(y),$$

and of expected continuation surplus

$$Z_t(b) = \int_{\underline{y}}^{\overline{y}} \int_{\epsilon} \max\left\{S_t(y, b) + \epsilon^R - \epsilon^A, 0\right\} \, dF_\epsilon dF^y = \int_{\underline{y}}^{\overline{y}} \alpha \ln\left(1 + e^{S_t(y,b)/\alpha}\right) \, dF^y.$$

The last equality comes from the fact that the difference of two Gumbel-distributed random variables follows a logistic distribution. The expected positive part of a logistic variable shifted by $S_t(y, b)$ corresponds to the softplus function evaluated at $S_t(y, b)$.

Proposition 3 (Monotonicity)**.** *In the above environment with $T < \infty$,*

1. **Values and Decision Rules across States** *For all t,*
   - 1.1 *$V_t^R(y, b)$ is strictly increasing in y and strictly decreasing in b.*
   - 1.2 *$W_t(b)$ is non-increasing in b.*
   - 1.3 *$\chi_t(b)$ is non-increasing in b.*
   - 1.4 *If $q_t(b')$ is non-increasing in b', $h_t(y, b)$ is non-increasing in y and non-decreasing in b.*
2. **Debt Prices across States:** *For all t, $q_t(b)$ is non-increasing in b.*
3. **Values and Default over Time:** *For all $t < T$ and all (y, b),*
   - 3.1 *$S_t(y, b) \geq S_{t+1}(y, b)$.*
   - 3.2 *$Z_t(b) \geq Z_{t+1}(b)$.*
   - 3.3 *$\chi_t(b) \geq \chi_{t+1}(b)$.*

The first two parts establish monotonicity of the repayment value in (y, b), as well as of the continuation value, repayment probability, and bond price in b for a given t. The proofs follow standard arguments. However, no general monotonicity result can be established for consumption. First, $h(y, b) - (1 - \lambda)b$

may be either positive (debt accumulation) or negative (debt buyback). Second, the budget set need not be convex when α is not large enough.

The third part shows that, for a given state (y, b), the repayment surplus and continuation surplus are weakly higher, and the default probability is weakly lower, the earlier is the period.[13] These results follow from a natural horizon argument: the sovereign cannot be worse off with a longer horizon, as it can always replicate the consumption plan of its future self. Assumption 2 is critical, as it ensures that average maturity remains constant over time, making such replication feasible.[14]

In contrast, no monotonicity result in t can be established for consumption, bond prices, or borrowing policies. Although default risk at a given (y, b) decreases with the horizon, bond prices need not increase due to dilution effects. Moreover, even if bond prices were monotone, borrowing policies h_t need not be, as optimal borrowing may increase or decrease in response to price changes. Since neither prices nor borrowing are monotone in t, consumption need not be monotone either.

Building on the monotonicity of the surplus $S_t(y, b)$ in t, the next proposition establishes existence and uniqueness of the equilibrium as the limit of equilibria in finite-horizon economies. The argument revolves around the joint convergence of the sequences of repayment surplus, bond policy and bond price. We focus on equilibria that arise as limits of finite-horizon problems under the assumptions that α is sufficiently large and that the repayment surplus $S_t(y, b)$ is monotone in t. Proposition 4 establishes uniqueness within this class of equilibria; it does not assert uniqueness of all Markov equilibria in the infinite-horizon model.

Proposition 4 (Existence and Uniqueness)**.** *Assume α is sufficiently large and $S_t(y, b) \geq S_{t+1}(y, b)$ for all $t < T$ and let $\{S_t, h_t, q_t\}_{t=0}^{\infty}$ denote the sequence generated by the recursive system:*

$$S_t(y, b) = \max_{b' \in \Gamma(y, b; q_t)} \left\{ u(y - b + q_t(b')[b' - (1-\lambda)b]) - u(y) + \beta \int \alpha \ln \left(1 + e^{S_{t+1}(y', b')/\alpha}\right) dF^y(y') \right\},$$

$$h_t(y, b) = \arg\max_{b' \in \Gamma(y, b; q_t)} \left\{ u(y - b + q_t(b')[b' - (1-\lambda)b]) - u(y) + \beta \int \alpha \ln \left(1 + e^{S_{t+1}(y', b')/\alpha}\right) dF^y(y') \right\},$$

$$q_t(b') = \bar{p} \int \phi_{t+1}(y', b') \left[1 + (1-\lambda) q_{t+1}(h_{t+1}(y', b'))\right] dF^y(y'),$$

---

[13] It can be shown that $V_t^R(y, b) > V_{t+1}^R(y, b)$ and $V_t^A(y) > V_{t+1}^A(y)$ for all $t < T$ and all (y, b) under the additional assumption that $u(\underline{c}) > 0$ where $\underline{c}$ is the lowest level of consumption achieved in equilibrium by adding a suitable large positive constant to the utility function.

[14] Suppose that Assumption 2 does not hold and the borrower only repays b in T. This implies that the debt issued in $T-1$ is one-period. If there is no default risk whatsoever, $q_{T-1}(b) = \bar{p}$ and $q_t(b) = q_{t+1}(b) + (\bar{p}(1-\lambda))^{T-t+1}$ for $t < T-1$. This implies that $q_t(b) \geq q_{t+1}(b)$ for all $t < T$ which can hinder the borrower to replicate buybacks operated by its future self under no default. This further prevents us to show monotonicity of the repayment surplus.

where $\phi_t(y', b') = e^{S_t(y',b')/\alpha}/(1 + e^{S_t(y',b')/\alpha})$. *Then the sequences* S_t, h_t, *and* q_t *converge uniformly to* S^∞, h^∞, *and* q^∞, *respectively. The limit tuple* $(S^\infty, h^\infty, q^\infty)$ *satisfies the associated fixed point system.*

The proposition establishes that the sequence $\{(S_t, h_t, q_t)\}$ converges uniformly to a fixed point $(S^\infty, h^\infty, q^\infty)$. The main difficulty in proving this result is that q_t is neither monotone in t nor independent of (S_t, h_t). If q_t were monotone in t, convergence would follow directly from the contraction property of the repayment problem for a given price sequence. If q_t were independent of h_t, convergence would follow from the monotonicity of S_t in t. Since neither condition holds, the proof proceeds by analyzing the entire recursive system (S_t, h_t, q_t). We first show that S_t converges uniformly to S^∞ by monotonicity in t. We then establish uniqueness of the limiting price function q^∞, which in turn pins down h^∞ uniquely by strong concavity of the objective. Uniform boundedness and equicontinuity of $\{q_t\}$ imply, via the Arzelà-Ascoli theorem, that every subsequence admits a uniformly convergent subsequence, and uniqueness of the fixed point limit implies the full sequence converges uniformly to q^∞. Uniform convergence of the bond policy sequence $h_t \to h^\infty$ follows immediately from continuity of the bond policy operator via Berge's maximum theorem.

The proof relies on uniform boundedness and equicontinuity of the bond price as well as on Berge's maximum theorem under unique maximizer. To guarantee this, α needs to be sufficiently large. This enables us to show that the objective function inherits the properties of the instantaneous utility function (see Proposition 2). Particularly strong concavity and L-smoothness are needed to show uniform boundedness and equicontinuity of h_t and q_t including their derivatives. Moreover, a large enough α convexifies the maximization problem guaranteeing a unique maximizer.

Our argument provides a selection mechanism that uniquely identifies the equilibrium $(S^\infty, h^\infty, q^\infty)$ as the limit of the recursive sequence $\{(S_t, h_t, q_t)\}$. In this sense, although global uniqueness of fixed points of the recursive system is not guaranteed, we obtain uniqueness of the selected equilibrium through convergent limits. The sequence $\{(S_t, h_t, q_t)\}$ identifies a unique and economically meaningful limit which corresponds to the limit of finite horizon economies. This unique equilibrium is a Markov equilibrium consistent with Definition 1. In other words, we rule out Markov-like equilibria that could be sustained by means of trigger strategies. Such equilibria can exist but do not correspond to the limit of finite-horizon economies.

It is useful to relate Proposition 4 to the multiplicity result of Aguiar and Amador (2020). They show that the long-term debt model admits two qualitatively distinct equilibria: a *borrowing* equilibrium, in which the sovereign enters the risky region with positive probability, and a *saving* equilibrium, in which the sovereign de-levers until default occurs with zero probability. Which of the two arises as the limit of finite-horizon economies depends on parameters — notably the maturity rate λ and the degree of impatience. Proposition 4 does not contradict this. Our claim is that, for any fixed primitives, the limit of finite-horizon equilibria is unique and corresponds to one of these two types — the type being itself a function of the primitives, not a free choice. The substantive content of our uniqueness result is therefore not the selection of borrowing versus saving (which is determined by primitives), but the exclusion of Markov-like equilibria sustained by trigger strategies in the sense of Krusell and Smith (2003), in which players coordinate on the history of past play even though the current state is fully payoff-relevant. Footnote 20 of Aguiar and Amador (2020), which warns that the finite-horizon limit cannot be used to consistently select a *type* of equilibrium, refers to the borrowing-vs-saving distinction; it does not preclude the use of the limit to select among Markov equilibria of a given type, which is the role played by our argument here. In the calibrated economies of Section 6 the equilibrium is of the borrowing type, satisfies Assumption 1, and is uniquely identified by the convergent sequence $\{(S_t, h_t, q_t)\}$.

5 The Generalized Euler Equation

Having shown existence and uniqueness, we now characterize the decision problem through the Generalized Euler Equation (GEE), that is the Euler equation which includes the derivative of bond prices, and through them the derivatives of future actions with respect to current actions.

5.1 Obtaining the Generalized Euler Equation

The GEE provides an important analytical tool to characterize the decision of the sovereign in cases where default occurs (in economies where there is no default the standard Euler equation applies). However, its derivation requires the bond price $q(b)$, the bond policy $h(y, b)$, and the repayment probability $\phi(y, b)$ to be differentiable in b. At this stage, we proceed by assuming that they are indeed differentiable and we will turn to prove it below in Section 5.2.

Besides differentiability, the statement that the GEE is a sufficient condition for optimality requires the decision problem to be convex. As shown in Proposition 2, this is only true when α is large enough. Without convexity, the first-order conditions are necessary but not sufficient. Concretely, there might be multiple local maxima in the GEE which complicates the use of such an analytical tool to characterize the equilibrium outcome. Hence, we assume that the scale parameter α is sufficiently large to ensure convexity of the problem. We discuss this assumption in Section 5.2. If the scale parameter α is not sufficiently large to ensure convexity the GEE still exists but will typically have more than one solution only one of which will be the global optimum. The GEE can still be used to characterize (and compute) the solution but we also have to keep track of the implied values to find out which of the solutions yields the global maximum.

Given that α is large enough and the differentiability of $q(b)$, $h(y,b)$ and $\phi(y,b)$, we can take the first-order condition of Equation (2) with respect to b',

$$\begin{aligned} u_c(\mathcal{C}(y,b,b'|q)) \Big[q(b') + q_b(b')\ (b' - (1-\lambda)b)\Big] = \\ \beta \int_{\underline{y}}^{\overline{y}} \phi(y',b')\ u_c(\mathcal{C}(y',b',h(y',b')|q))\ [1 + (1-\lambda)\ q(h(y',b'))]\ dF^y, \end{aligned}$$

where $\mathcal{C}(y,b,b'|q) \equiv y - b + q(b')(b' - (1-\lambda)b)$ corresponds to consumption today and $\mathcal{C}(y',b',h(y',b')|q)$ to consumption tomorrow. This notation makes explicit the functional dependencies. In addition, $q_b(b')$ is the derivative of the bond price with respect to b'. The left-hand side of the above expression represents the marginal benefit of one additional unit of debt while the right-hand side corresponds to the marginal cost. The marginal benefit is the consumption gain from marginal borrowing taking into account the impact it has on the price of the debt. The marginal cost of an additional unit of borrowing is the expected marginal utility loss of paying the coupon and rolling over unmatured debt at tomorrow's price in repayment states.

For a given state (y,b) and borrowing b', the first-order condition contains the functions ϕ, h, q and q_b. Since we seek a GEE that involves decision rules only, and not equilibrium price functions, one step is the substitution of the price derivative q_b. Taking the first derivative of the price of debt as described in Equation (5) gives

$$\begin{aligned} q_b(b') =& \overline{p}(1-\lambda) \int_{\underline{y}}^{\overline{y}} \phi(y',b')\ q_b\left[h(y',b')\right]\ h_b(y',b')\ dF^y \\ &+ \overline{p} \int_{\underline{y}}^{\overline{y}} \phi_b(y',b') \left[1 + (1-\lambda)\ q(h(y',b'))\right]\ dF^y. \end{aligned} \tag{6}$$

where $h_b(y',b')$ and $\phi_b(y',b')$ denote the first derivative of the bond policy and the repayment probability, respectively. The first term with the integral is the dilution risk, while the second term gives the loss of value per unit of debt weighted by the marginal probability of default. The dilution term of the price derivative is therefore itself represented by the future price derivatives $q_b(h(y',b'))$. In fact, we can use the value of q_b implied by the first-order condition to get an expression that does not depend on future derivatives. Inverting the first-order condition, in an equilibrium where $b' = h(y,b)$, we can write the price derivatives as $q_b(h(y,b)) = \mathcal{B}(y,b|h,\phi,q)$ where $\mathcal{B}$ is a known expression given by

$$\mathcal{B}(y,b|h,\phi,q) \;=\; \frac{\beta \int_{\underline{y}}^{\overline{y}} \phi(y',h(y,b))\; u_c(c')\; [1+(1-\lambda)q(h(y',h(y,b)))]\; dF^y - u_c(c)q(h(y,b))}{u_c(c)\; [h(y,b)-(1-\lambda)b]},$$

with $c = \mathcal{C}(y,b,h(y,b)|q)$ and $c' = \mathcal{C}[y',h(y,b),h(y',h(y,b))|q]$.[15]

Since the present self assumes equilibrium in future, the future derivatives can analogously be expressed so that $q_b(h(y',b')) = \mathcal{B}(y',b'|h,\phi,q)$, and the current price derivative, as a function of b', becomes

$$q_b(b') = \overline{p}(1-\lambda)\int_{\underline{y}}^{\overline{y}} \phi(y',b')\; \mathcal{B}(y',b'|h,\phi,q)\; h_b(y',b')\; dF^y \\ + \overline{p}\int_{\underline{y}}^{\overline{y}} \phi_b(y',b')\left[1+(1-\lambda)\; q(h(y',b'))\right]\; dF^y.$$

Combining the above expressions gives the GEE. Formally,

$$\begin{aligned} u_c(c)\Bigg[q(b') + (b'-(1-\lambda)b)\; \Bigg\{\overline{p}(1-\lambda)\int_{\underline{y}}^{\overline{y}} \phi(y',b')\; \mathcal{B}(y',b'|h,\phi,q)\; h_b(y',b')\; dF^y \\ + \overline{p}\int_{\underline{y}}^{\overline{y}} \phi_b(y',b')\; [1+(1-\lambda)\; q(h(y',b'))]\; dF^y\Bigg\}\Bigg] \\ = \beta\int_{\underline{y}}^{\overline{y}} \phi(y',b')\; u_c(c')\; [1+(1-\lambda)\; q(h(y',b'))]\; dF^y. \qquad \text{(GEE)} \end{aligned}$$

with $c = \mathcal{C}(y,b,h(y,b)|q)$ and $c' = \mathcal{C}[y',h(y,b),h(y',h(y,b))|q]$.

The (GEE) still contains the price function q. Yet Equation (5) fully characterizes that price function q for given decision rules ϕ and h. Note then how substituting q from (GEE) yields a formula that is only a function of current and future decision rules without any need of involving markets. It is then a characterization that only involves the game against future selves, albeit all future decision rules, so using the pricing function q provides a convenient way to simplify notation. The GEE thus highlights that borrowing decisions internalize both default risk and dilution through price derivatives, providing a transparent decomposition of the forces shaping the sovereign's choice.

Under Assumption 1, the equilibrium debt policy is strictly positive, $h_t(y,b) > 0$, so the GEE

---

[15] The expression $\mathcal{B}$ has $h(y,b)-(1-\lambda)b$ in the denominator, which is zero when the sovereign exactly rolls over for some boundary value of y ($b' = (1-\lambda)b$). Since the price derivative becomes irrelevant for the optimality condition in this case, the numerator also becomes zero and so does $\mathcal{B}$.

characterizes the optimum as an equality at every (y, b). The choice set itself remains the closed interval $B = [0, \overline{b}]$; strict positivity is a property of the selected equilibrium, not of the feasibility set. If we considered non-interior borrowing choice $b' = 0$, then the GEE would hold as a weak inequality. Since we consider environments in which the borrower enters the risky region (b' such that default occurs with positive probability), the optimal borrowing solves the full (GEE) with both dilution and default risk. If we considered cases in which the borrower remains in the risk-free region, the dilution and default terms would vanish and the GEE would reduce to the standard Euler equation

$$u_c(c)\ q(b') = \beta \int_{\underline{y}}^{\overline{y}} u_c(c')\ [1 + (1-\lambda)\ q(h(y', b'))]\ \ dF^y, \tag{EE}$$

with $c = \mathcal{C}(y, b, h(y, b)|q)$ and $c' = \mathcal{C}[y', h(y, b), h(y', h(y, b))|q]$.[16]

5.2 Convexity and Differentiability

The use of the GEE as a sufficient condition to characterize optimality critically depends on two elements: the convexity of the decision problem and the differentiability of q, ϕ and h.

The first-order conditions are necessary and sufficient for a maximum under a concave objective function and a convex budget set. The introduction of extreme value shocks addresses this issue as it allows agents to randomize over decisions. In each $b \in B$, there is always a positive probability that the default option is chosen. Such probability distribution can convexify the decision problem. Nevertheless, the mere introduction of randomization is usually not sufficient and one needs to ensure that randomization is intense enough as shown in Proposition 2.

Regarding differentiability, as already said in Section 2.3, in general there may be two forms of discontinuity: kinks and jumps. Kinks would not represent an issue as integration over kinks preserve not only continuity but also differentiability. In contrast, jump discontinuities would be problematic as the integral is continuous but not differentiable at this point. We got rid of potential jumps at b^* with the continuous repayment probability obtained with extreme value shocks.

With long-term debt, the bond price depends on the optimal policy h, so differentiability of the bond price has to be established recursively through the finite horizon problem presented in Section 4.

---

[16] As discussed in Section 2.3, without extreme value shocks, there may be three regions given the threshold values b^r and b^*. In points such that $b' \in [0, b^r]$, the optimal borrowing is the solution to the EE. In points such that $b' \in (b^r, b^*]$, the optimal borrowing is the solution to the GEE with dilution risk and without default risk. Finally, in points such that $b' > b^*$, the optimal borrowing is the solution to the GEE with both dilution and default risk. There is a further corner solution at $b' = b^*$ where the GEE is not satisfied.

Proposition 5 (Smoothness). *If Assumption 1 holds, i.e.* $h_t(y, b) > 0$ *for all* (y, b) *and* $t < T$, ϕ_t, h_t *and* q_t *are of class* $\mathcal{C}^\infty$ *for all* $t \leq T$.

The proposition relies on an inductive argument. At time T, the bond policy function only depends on the logit repayment probability. Hence, it is obviously of class $\mathcal{C}^\infty$. Given this, at time $T-1$ we can show that the GEE is itself of class $\mathcal{C}^\infty$. By application of the implicit function theorem, the bond policy at $T-1$ inherits the properties of the GEE. Given this, q_{T-2} is also of class $\mathcal{C}^\infty$. We then repeat this argument backward until $t = 0$.

The fact that we rely on $\mathcal{C}^\infty$ is crucial as the proof works recursively through the finite horizon problem. If for some $0 < t < T$, the bond policy function h_t were of class $\mathcal{C}^k$ for $k < \infty$, then h_{t-1} would be of class $\mathcal{C}^{k-1}$ since the GEE at time $t-1$ depends on $h_{b,t}$ which is of class $\mathcal{C}^{k-1}$. Hence, if $k < \infty$, one loses one degree of differentiability for each iteration. This could eventually prevent to show that ϕ, h and q are differentiable.

Once the differentiability of ϕ, h and q has been established, we need to show that the derivatives converge to the appropriate policies.

Proposition 6 (Limiting Derivatives). *Let* $\{h_{b,t}\}$ *and* $\{q_{b,t}\}$ *denote the sequence of derivatives of* $\{h_t\}$ *and* $\{q_t\}$, *respectively. Under Assumption 1,* $h_t \to h^\infty$ *uniformly on* $Y \times B$ *and* $q_t \to q^\infty$ *uniformly on* $Y \times B$ *and large enough* α, *it holds that* $h_{b,t} \to \frac{d}{db}h^\infty$ *uniformly on* $Y \times B$ *and* $q_{b,t} \to \frac{d}{db}q^\infty$ *uniformly on* $Y \times B$.

For the derivative of the bond policy, we rely on the implicit function theorem stating that this derivative is a function of the second derivatives of the objective function. The convergence argument builds on the fact that the objective function's second and third derivatives are bounded given the assumption of strong concavity and L-smoothness. This enables us to show equicontinuity and apply the Arzelà-Ascoli theorem which ensures existence of a convergent subsequence. The full sequence convergence then follows from the fact that h_t is differentiable and converges uniformly. For the bond price, we use the same argument since the derivatives of q_t are bounded for all t when α is large enough.

6 Computation

We now demonstrate the computational advantages of the GEE characterization developed in the previous sections. We apply our methods to a fully calibrated version of the model following Chatterjee and Eyigungor (2012), incorporating persistent income, long-term debt with coupon payments, stochastic market re-entry, and output penalties upon default. The calibrated application already exercises two natural extensions of the baseline — stochastic re-entry from default via the probability ξ and a non-i.i.d. AR(1) endowment — and a third, a positive recovery rate $\eta \in [0,1]$, is treated analytically in Appendix B.3, where the GEE extends with a modified pricing coefficient and, when the recovery is financed by the defaulter (an exogenous-size partial default), an additive term.

The main text compares three solution methods. First, the discrete-state value function iteration (VFI) that solves the sovereign's problem by discretizing both the state and action spaces, transforming the problem into one of finite dimension, and iterating backward from the terminal period to the present. It is very robust but accuracy requires a large number of debt-choice points to avoid solutions being driven by grid rounding rather than by maximization. Second, policy function iteration (PI) that exploits the GEE that we have derived here. It approximates the policy function with a low-order Chebyshev basis and uses bisection to solve for the optimal borrowing choice. Third, we use the endogenous grid method (EGM) to exploit the GEE taking advantage that the consumption choice as function of borrowing has a closed-form solution when risk aversion is 2. A fourth method, that approximates the value function (and iterates on it) using Chebyshev interpolation, is described and benchmarked separately in Appendix C.[17]

6.1 Model and Calibration

Denote by $F^y(y'|y)$ the conditional distribution function of y' given y, by $\xi \in [0,1]$ the re-entry probability, and by $\varphi(y)$ the endowment loss upon default. The value under financial autarky is

$$V^A(y) = u\left[y - \varphi(y)\right] + \beta \int_{\underline{y}}^{\overline{y}} \left[(1-\xi)\ V^A(y') + \xi\ V^R(y',0)\right]\ dF^y(y'|y).$$

The value under repayment is

$$V^R(y,b) = \max_{b'}\ u\left\{y - \left[\lambda + (1-\lambda)z\right]b + q(y,b')\left[b' - (1-\lambda)b\right]\right\} + \beta W(y,b').$$

The inclusive continuation value is

$$W(y,b') = \int_{\underline{y}}^{\overline{y}} \alpha \log\left[\exp\left(\frac{V^R(y',b')}{\alpha}\right) + \exp\left(\frac{V^A(y')}{\alpha}\right)\right] dF^y(y'|y).$$

---

[17] There are other approaches in the literature. Jang and Lee (2024) use the Endogenous Grid Method for a sovereign default problem with short-term debt, implicitly assuming differentiability. Kiiashko and Maliar (2021) in independent work use the Endogenous Grid Method with long-term debt, adding an i.i.d. normal shock to the value of default and computing numerically the derivatives of both the value function and the pricing function. The numerical computation of the derivatives avoids using the information that our characterization of the GEE provides. See Mihalache (2025) for a recent survey of computational methods for sovereign default models, which include extending the use of Extreme Value Shocks to also characterize intensive-margin borrowing choices.

The repayment probability is

$$\phi\left(y,b\right)=\frac{\exp\left(V^R\left(y,b\right)/\alpha\right)}{\exp\left(V^R\left(y,b\right)/\alpha\right)+\exp\left(V^A\left(y\right)/\alpha\right)}.$$

Given this repayment probability, the price of one unit of long-term debt is

$$q\left(y,b'\right)=\overline{p}\int_{\underline{y}}^{\overline{y}}\phi\left(y',b'\right)\left[\lambda+\left(1-\lambda\right)z+\left(1-\lambda\right)q\left(y',h\left(y',b'\right)\right)\right]dF^y\left(y'|y\right),$$

where $\overline{p}=1/\left(1+r\right)$. The dependence of the bond price on current income y comes from the persistence of the income process.

Relative to the abstract environment of Section 2, the calibrated model carries an explicit coupon z; the budget term $[\lambda+(1-\lambda)z]b$ reduces to b under the normalization $z=1$ used in earlier sections (without loss of generality by rescaling debt by $[\lambda+(1-\lambda)z]$).

We adopt the calibration strategy and details of Chatterjee and Eyigungor (2012). Utility is CRRA,

$$u\left(c\right)=\left(1-\beta\right)\frac{c^{1-\sigma}}{1-\sigma},\qquad\sigma=2.$$

The output penalty upon default is

$$\varphi\left(y\right)=\max\left\{0,d_0y+d_1y^2\right\}.$$

The endowment process is

$$\log y_t=\rho\log y_{t-1}+u_t,\qquad u_t\sim N\left(0,\sigma_u^2\right).$$

Chatterjee and Eyigungor (2012) use the mean and standard deviation of spreads and the debt-to-output ratio as calibration targets, and the discount rate and the linear and quadratic default cost coefficients as parameters. We add the correlation between spreads and output as an additional target and the scale of the extreme-value shocks, α, as an additional parameter. We use the PI method to implement the calibration minimizing the distance between model-generated and target moments. Table 1 reports the parameter values.

Table 1: Parameter values.

Parameter	Value	Description
Panel A. Set ex ante (following Chatterjee and Eyigungor (2012)).		
σ	2	Risk aversion
r	0.01	Risk-free return
λ	0.05	Reciprocal of average maturity
z	0.03	Coupon payments
ξ	0.0385	Probability of reentry
ρ	0.948503	Autocorrelation of log y
σ_u	0.027092	Standard deviation of u
Panel B. Calibrated internally.		
β	0.962	Discount factor
d_0	-0.1606	Default cost parameter (linear)
d_1	0.2365	Default cost parameter (quadratic)
α	0.0039	Scale of extreme-value taste shocks

6.2 Methods

The solution method commonly referred to as value function iteration (VFI) uses the following set of functional equations:

$$h(y,b) = \arg\max_{b'} \left\{ u\left(y - [\lambda + (1-\lambda) z] b + q(y,b') \left[b' - (1-\lambda) b\right]\right) + \beta W(y,b') \right\}, \tag{7}$$

$$V^R(y,b) = u(y - [\lambda + (1-\lambda) z] b + q(y,h(y,b)) [h(y,b) - (1-\lambda) b]) + \beta W(y,h(y,b)), \tag{8}$$

$$W(y,b') = \int_{\underline{y}}^{\overline{y}} \alpha \log\left[\exp\left(\frac{V^R(y',b')}{\alpha}\right) + \exp\left(\frac{V^A(y')}{\alpha}\right)\right] dF^y(y'|y), \tag{9}$$

$$q(y,b') = \overline{p} \int_{\underline{y}}^{\overline{y}} \phi(y',b') \left[\lambda + (1-\lambda) z + (1-\lambda) q(y',h(y',b'))\right] dF^y(y'|y), \tag{10}$$

$$\phi(y,b) = \frac{\exp\left(V^R(y,b)/\alpha\right)}{\exp(V^R(y,b)/\alpha) + \exp(V^A(y)/\alpha)}. \tag{11}$$

The left hand side of this system is the set of unknown functions $\left\{h, V^R, W, q, \phi\right\}$.

Methods described as policy function iteration (PI and EGM) use first-order conditions, which in our case imply the GEE. They do not require discretizing the choice set. These methods use Equations (8) to (11), replace Equation (7) by the GEE in Equation (12), and add derivative equations for the continuation and pricing objects:

$$0 = u_c\left(c\left(y, b, h\left(y, b\right)\right)\right)\left[q\left(y, h\left(y, b\right)\right) + q_b\left(y, h\left(y, b\right)\right)\left[h\left(y, b\right) - \left(1-\lambda\right) b\right]\right] + \beta W_b\left(y, h\left(y, b\right)\right), \tag{12}$$

$$\begin{aligned}
W_b\left(y, b'\right) &= -\int_{\underline{y}}^{\overline{y}} \phi\left(y', b'\right) u_c\left(c\left(y', b', h\left(y', b'\right)\right)\right)\left[\lambda + \left(1-\lambda\right) z + \left(1-\lambda\right) q\left(y', h\left(y', b'\right)\right)\right] dF^y\left(y'|y\right), \\
q_b\left(y, b'\right) &= \overline{p}\int_{\underline{y}}^{\overline{y}} \phi_b\left(y', b'\right)\left[\lambda + \left(1-\lambda\right) z + \left(1-\lambda\right) q\left(y', h\left(y', b'\right)\right)\right] dF^y\left(y'|y\right) \\
&\quad + \overline{p}\left(1-\lambda\right)\int_{\underline{y}}^{\overline{y}} \phi\left(y', b'\right) q_b\left(y', h\left(y', b'\right)\right) h_b\left(y', b'\right) dF^y\left(y'|y\right), \qquad (13) \\
\phi_b\left(y, b\right) &= -\frac{1}{\alpha} u_c\left(c\left(y, b, h\left(y, b\right)\right)\right)\left[\lambda + \left(1-\lambda\right) z + \left(1-\lambda\right) q\left(y, h\left(y, b\right)\right)\right]\phi\left(y, b\right)\left[1 - \phi\left(y, b\right)\right],
\end{aligned}$$

where

$$c\left(y, b, b'\right) = y - \left[\lambda + \left(1-\lambda\right) z\right] b + q\left(y, b'\right)\left[b' - \left(1-\lambda\right) b\right].$$

There are four additional unknowns, $\{q_b, W_b, \phi_b, h_b\}$, bringing the total to nine (h, V^R, W, q, ϕ from the VFI block, plus the four derivatives). The four equations above for the GEE, W_b, q_b, and ϕ_b close all but one: the system does not pin down h_b directly. In the PI implementation we therefore do not solve for h_b from an analytical equation. Instead, the GEE is solved pointwise by root-finding for a raw policy on the fixed debt grid. Finite-differencing this raw policy directly proved unstable on finer grids, so we project it row by row in income onto a Chebyshev basis in b and read h_b off as the derivative of the projected policy. The projection serves two roles: it supplies the missing h_b, and it damps the local root-finding irregularities on fine debt grids that would otherwise be amplified through the price-derivative Equation (13). The EGM implementation needs no such step: as described below, the endogenous-grid construction already interpolates the policy back onto the fixed debt grid to recover $h\left(y, b\right)$, and h_b is obtained by finite-differencing that interpolated policy — the very interpolation the method performs anyway, with no additional smoothing.

The monotonicity of the borrowing policy $b' = h\left(y, b\right)$ implies that h is invertible in b, so that there exists a function $b = g\left(y, b'\right)$. This provides the theoretical basis for using the Endogenous Grid Method. In general, solving the GEE (*Equation* (12)) for b' requires a numerical solver. However, in the case of CRRA utility with $\sigma = 2$, consumption can be obtained as the solution to a second-order polynomial.[18]

This closed-form solution for consumption as a function of (y, b') is what enables the endogenous grid method to avoid root-finding entirely.

We now provide detailed pseudocode for the three main-text methods we compare: discrete-state VFI, PI with GEE, and EGM with GEE. The fourth method, VFI with Chebyshev interpolation, is presented in Appendix C. All methods iterate backward through a finite-horizon version of the problem until convergence to the infinite-horizon solution, checking the sup-norm of successive iterates of prices and values.

Algorithm 1: Discrete State Space Value Function Iteration. The discrete state space VFI method solves the Bellman system directly on a finite tensor grid. Let $\mathcal{Y} = \left\{y_1, \dots, y_{N_y}\right\}$ denote the income grid and $\mathcal{B} = \left\{b_1, \dots, b_{N_b}\right\}$ denote the debt grid. The same grid $\mathcal{B}$ is used for inherited debt and for next-period debt choices. Let $\pi_{j\ell}$ denote the probability of moving from current income y_j to next-period

---

[18] Using the budget constraint to solve for current debt b gives

$$b = \frac{y + q\left(y, b'\right) b' - c}{\lambda + (1-\lambda) z + (1-\lambda) q\left(y, b'\right)}.$$

Substituting this expression into the GEE and rearranging yields

$$\begin{aligned}
&\beta W_b\left(y, b'\right) c^2 + (1-\beta) \frac{q_b\left(y, b'\right)(1-\lambda)}{\lambda + (1-\lambda) z + (1-\lambda) q\left(y, b'\right)} c \\
&+ (1-\beta)\left\{q\left(y, b'\right) + q_b\left(y, b'\right) b' - (1-\lambda) q_b\left(y, b'\right)\left[\frac{y + q\left(y, b'\right) b'}{\lambda + (1-\lambda) z + (1-\lambda) q\left(y, b'\right)}\right]\right\} = 0.
\end{aligned}$$

This is a quadratic equation for c. The solution is given by

$$\begin{aligned}
c &= \frac{-a_2 \pm \sqrt{a_2^2 - 4a_1 a_3}}{2a_1}, \\
a_1 &= \beta W_b\left(y, b'\right) < 0, \\
a_2 &= (1-\beta) \frac{q_b\left(y, b'\right)(1-\lambda)}{\lambda + (1-\lambda) z + (1-\lambda) q\left(y, b'\right)} < 0, \\
a_3 &= (1-\beta)\left\{q\left(y, b'\right) + q_b\left(y, b'\right) b' - (1-\lambda) q_b\left(y, b'\right)\left[\frac{y + q\left(y, b'\right) b'}{\lambda + (1-\lambda) z + (1-\lambda) q\left(y, b'\right)}\right]\right\} > 0.
\end{aligned}$$

Note there are possibly two roots to this equation; however, we know it is optimal to choose the highest level of c, and given that $\sqrt{a_2^2 - 4a_1 a_3} > a_2$ and $a_1 < 0$, the solution must be

$$c = \frac{-a_2 - \sqrt{a_2^2 - 4a_1 a_3}}{2a_1}.$$

Note that $a_3 > 0$ in the calibrated equilibrium is verified state by state on the validation grid; together with $a_1 < 0$ this ensures the discriminant $a_2^2 - 4a_1 a_3 \geq 0$ and that the larger root corresponds to the positive-consumption solution; see the diagnostic logged by the `egm` routine in the replication package.

income y_ℓ. Given a price schedule q, define

$$c\left(y_j, b_i, b_k'; q\right) = y_j - \left[\lambda + (1-\lambda) z\right] b_i + q\left(y_j, b_k'\right)\left[b_k' - (1-\lambda) b_i\right].$$

Step 1. Initialize. Set $n = 0$. Choose initial guesses for $q^{(0)}(y, b')$, $W^{(0)}(y, b')$, $V^{R,(0)}(y, b)$, and $V^{A,(0)}(y)$ on the grid. Construct

$$\phi^{(0)}(y_j, b_i) = \frac{\exp\left(V^{R,(0)}(y_j, b_i)/\alpha\right)}{\exp\left(V^{R,(0)}(y_j, b_i)/\alpha\right) + \exp\left(V^{A,(0)}(y_j)/\alpha\right)}.$$

Step 2. Maximize over the discrete choice set. For each (y_j, b_i), choose

$$h^{(n+1)}(y_j, b_i) \in \arg\max_{b_k' \in \mathcal{B}} \left\{ u\left(c\left(y_j, b_i, b_k'; q^{(n)}\right)\right) + \beta W^{(n)}(y_j, b_k') \right\},$$

treating choices with non-positive consumption as infeasible. The maximized objective defines $V^{R,(n+1)}(y_j, b_i)$.

Step 3. Update autarky. Update

$$V^{A,(n+1)}(y_j) = u\left(y_j - \varphi(y_j)\right) + \beta \sum_{\ell=1}^{N_y} \pi_{j\ell} \left[(1-\xi) V^{A,(n)}(y_\ell) + \xi V^{R,(n)}(y_\ell, 0)\right].$$

Step 4. Update repayment probabilities and continuation values. Set

$$\phi^{(n+1)}(y_j, b_i) = \frac{\exp\left(V^{R,(n+1)}(y_j, b_i)/\alpha\right)}{\exp\left(V^{R,(n+1)}(y_j, b_i)/\alpha\right) + \exp\left(V^{A,(n+1)}(y_j)/\alpha\right)},$$

and update

$$W^{(n+1)}(y_j, b_k') = \sum_{\ell=1}^{N_y} \pi_{j\ell} \alpha \log\left[\exp\left(\frac{V^{R,(n+1)}(y_\ell, b_k')}{\alpha}\right) + \exp\left(\frac{V^{A,(n+1)}(y_\ell)}{\alpha}\right)\right].$$

Step 5. Update prices. For each (y_j, b_k'), set

$$q^{(n+1)}(y_j, b_k') = \bar{p} \sum_{\ell=1}^{N_y} \pi_{j\ell} \phi^{(n+1)}(y_\ell, b_k') \left[\lambda + (1-\lambda) z + (1-\lambda) q^{(n)}\left(y_\ell, h^{(n+1)}(y_\ell, b_k')\right)\right].$$

Because $\mathcal{B}$ is also the choice grid, the future policy $h^{(n+1)}(y_\ell, b_k')$ is a grid point, so the final price term is evaluated by lookup.

Step 6. Check convergence. If

$$\left\| q^{(n+1)} - q^{(n)} \right\|_\infty < \texttt{tol} \qquad \text{and} \qquad \left\| V^{R,(n+1)} - V^{R,(n)} \right\|_\infty < \texttt{tol},$$

stop. Otherwise, set $n \leftarrow n+1$ and return to Step 2.

This method is the most direct benchmark: it solves the Bellman problem by exhaustive search over a finite action set. Its main drawback is that accurate policies require a fine debt grid, since the optimal choice is constrained to lie on the grid. This makes the method substantially more expensive than the GEE-based algorithms as the debt grid is refined.

Algorithm 2: Policy Function Iteration with GEE. The PI method replaces the discrete maximization step with the GEE, solved pointwise for the next-period debt choice b' on the exogenous current-debt grid; it does not use the endogenous-grid inversion introduced in Algorithm 3 below. Let $\mathcal{B} = \{b_1, \dots, b_{N_b}\}$ and $\mathcal{Y} = \left\{y_1, \dots, y_{N_y}\right\}$ denote the debt and income grids, and let $\pi_{j\ell}$ denote the transition probability from y_j to y_ℓ. Off-grid evaluations in the debt dimension use shape preserving cubic hermite interpolation: q is interpolated using (q, q_b), and W is interpolated using (W, W_b). For compactness, define

$$c\left(y, b, b'; q\right) = y - \left[\lambda + (1-\lambda) z\right] b + q\left(y, b'\right)\left[b' - (1-\lambda) b\right].$$

Step 1. Initialize. Set $n = 0$. Choose initial guesses for

$$q^{(0)}, \quad q_b^{(0)}, \quad W^{(0)}, \quad W_b^{(0)}, \quad V^{R,(0)}, \quad V^{A,(0)}$$

on the relevant grids, and construct $\phi^{(0)}$ from *Equation* (11).

Step 2. Solve the GEE for a raw policy update. Given $\left\{q^{(n)}, q_b^{(n)}, W_b^{(n)}\right\}$, solve, for each (y_j, b_i),

$$0 = u_c\left(c\left(y_j, b_i, b'; q^{(n)}\right)\right)\left[q^{(n)}\left(y_j, b'\right) + q_b^{(n)}\left(y_j, b'\right)\left[b' - (1-\lambda) b_i\right]\right] + \beta W_b^{(n)}\left(y_j, b'\right)$$

over $b' \in [0, b_{\max}]$, using bisection. Denote the resulting policy by $h_{\text{raw}}^{(n+1)}\left(y_j, b_i\right)$.

Step 3. Project the policy update. For each y_j, project $h_{\text{raw}}^{(n+1)}\left(y_j, \cdot\right)$ onto a low-order Chebyshev basis in b:

$$h^{(n+1)}\left(y_j, b\right) = \sum_{k=0}^{K_h} a_{jk}^{(n+1)} T_k\left(2\frac{b - b_{\min}}{b_{\max} - b_{\min}} - 1\right).$$

The derivative $h_b^{(n+1)}$ is computed by analytically differentiating this projected policy. The projection smooths local root-finding irregularities before they enter the value, price, and derivative updates.

Step 4. Update values and repayment probabilities. Using the projected policy, compute

$$c_{ji}^{(n+1)} = c\left(y_j, b_i, h^{(n+1)}\left(y_j, b_i\right); q^{(n)}\right),$$

and update

$$V^{R,(n+1)}\left(y_j, b_i\right) = u\left(c_{ji}^{(n+1)}\right) + \beta W^{(n)}\left(y_j, h^{(n+1)}\left(y_j, b_i\right)\right),$$

$$V^{A,(n+1)}\left(y_j\right) = u\left(y_j - \varphi\left(y_j\right)\right) + \beta \sum_{\ell=1}^{N_y} \pi_{j\ell}\left[\left(1-\xi\right) V^{A,(n)}\left(y_\ell\right) + \xi V^{R,(n)}\left(y_\ell, 0\right)\right].$$

Then update $\phi^{(n+1)}$ from *Equation* (11) and $W^{(n+1)}$ from *Equation* (9).

Step 5. Update derivatives and prices. Define the long-term payoff term

$$\Gamma_{\ell i}^{(n+1)} = \lambda + (1-\lambda) z + (1-\lambda) q^{(n)}\left(y_\ell, h^{(n+1)}\left(y_\ell, b_i\right)\right).$$

Then update

$$\phi_b^{(n+1)}\left(y_j, b_i\right) = -\frac{1}{\alpha} u_c\left(c_{ji}^{(n+1)}\right) \Gamma_{ji}^{(n+1)} \phi^{(n+1)}\left(y_j, b_i\right)\left[1 - \phi^{(n+1)}\left(y_j, b_i\right)\right],$$

$$W_b^{(n+1)}\left(y_j, b_i\right) = -\sum_{\ell=1}^{N_y} \pi_{j\ell} \phi^{(n+1)}\left(y_\ell, b_i\right) u_c\left(c_{\ell i}^{(n+1)}\right) \Gamma_{\ell i}^{(n+1)},$$

$$\begin{aligned} q_b^{(n+1)}\left(y_j, b_i\right) = {} & \bar{p} \sum_{\ell=1}^{N_y} \pi_{j\ell} \phi_b^{(n+1)}\left(y_\ell, b_i\right) \Gamma_{\ell i}^{(n+1)} \\ & + \bar{p}\left(1-\lambda\right) \sum_{\ell=1}^{N_y} \pi_{j\ell} \phi^{(n+1)}\left(y_\ell, b_i\right) q_b^{(n)}\left(y_\ell, h^{(n+1)}\left(y_\ell, b_i\right)\right) h_b^{(n+1)}\left(y_\ell, b_i\right), \end{aligned}$$

$$q^{(n+1)}\left(y_j, b_i\right) = \bar{p} \sum_{\ell=1}^{N_y} \pi_{j\ell} \phi^{(n+1)}\left(y_\ell, b_i\right) \Gamma_{\ell i}^{(n+1)}.$$

Here $q^{(n)}$ and $q_b^{(n)}$ are evaluated off grid by piecewise cubic hermite interpolation.

Step 6. Check convergence. If

$$\left\| q^{(n+1)} - q^{(n)} \right\|_\infty < \texttt{tol} \qquad \text{and} \qquad \left\| V^{R,(n+1)} - V^{R,(n)} \right\|_\infty < \texttt{tol},$$

stop. Otherwise, set $n \leftarrow n+1$ and return to Step 2.

Algorithm 3: Endogenous Grid Method (EGM) with GEE. When $\sigma = 2$, the GEE can be rearranged into a quadratic equation for consumption as a function of (y, b'). The EGM uses this closed-form solution to avoid the state-by-state root-finding step in Algorithm 2. Let $\mathcal{B} = \{b_1, \dots, b_{N_b}\}$ denote the exogenous grid for current debt, and let $\mathcal{B}' = \left\{b_1', \dots, b_{N_{b'}}'\right\}$ denote the exogenous grid for next-period debt choices. Let $\mathcal{Y} = \left\{y_1, \dots, y_{N_y}\right\}$ denote the income grid.

Step 1. Initialize. Set $n = 0$. Choose initial guesses for

$$q^{(0)}, \quad q_b^{(0)}, \quad W^{(0)}, \quad W_b^{(0)}, \quad V^{R,(0)}, \quad V^{A,(0)}$$

on the relevant grids, and construct $\phi^{(0)}$ from *Equation* (11).

Step 2. Solve the GEE on the exogenous b' grid. For each $(y_j, b'_k) \in \mathcal{Y} \times \mathcal{B}'$, evaluate $q^{(n)}(y_j, b'_k)$, $q_b^{(n)}(y_j, b'_k)$, and $W_b^{(n)}(y_j, b'_k)$ by interpolation, and solve the GEE for the feasible consumption level $c^{(n+1)}(y_j, b'_k)$, which at $\sigma = 2$ is given in closed form by the quadratic formula implied by Equation (12). This step replaces the bisection over b' used in Algorithm 2.

Step 3. Recover the endogenous current-debt grid. Given $c^{(n+1)}(y_j, b'_k)$, recover the current debt level consistent with the budget constraint:

$$b^{(n+1)}(y_j, b'_k) = \frac{y_j + q^{(n)}(y_j, b'_k)\, b'_k - c^{(n+1)}(y_j, b'_k)}{\lambda + (1-\lambda)\, z + (1-\lambda)\, q^{(n)}(y_j, b'_k)}.$$

For each y_j, the pairs

$$\left(b^{(n+1)}(y_j, b'_k), b'_k\right)_{k=1}^{N_{b'}}$$

define the policy on an endogenous grid for current debt.

Step 4. Interpolate onto the exogenous current-debt grid. For each income state y_j, interpolate the mapping

$$b^{(n+1)}(y_j, b'_k) \mapsto b'_k$$

onto $\mathcal{B}$ to obtain $h^{(n+1)}(y_j, b_i)$. Obtain $h_b^{(n+1)}$ by finite-differencing this interpolated policy along the debt grid. Because the endogenous-grid construction already performs this interpolation, EGM recovers the derivative without the separate Chebyshev projection step that PI requires.

Step 5. Apply the common GEE update. Given $h^{(n+1)}$ and $h_b^{(n+1)}$, update V^R, V^A, ϕ, W, W_b, q_b, and q exactly as in the value, derivative, and price-update steps of Algorithm 2.

Step 6. Check convergence. If

$$\left\| q^{(n+1)} - q^{(n)} \right\|_\infty < \texttt{tol} \qquad \text{and} \qquad \left\| V^{R,(n+1)} - V^{R,(n)} \right\|_\infty < \texttt{tol},$$

stop. Otherwise, set $n \leftarrow n + 1$ and return to Step 2.

Relative to PI, the only substantive change is the policy update. PI solves the GEE directly for b' at each current state (y, b). EGM instead chooses an exogenous grid for b', reads the corresponding consumption from the closed-form quadratic available at $\sigma = 2$, recovers the implied current debt b from the budget constraint, and interpolates the resulting endogenous grid back onto $\mathcal{B}$. The closed form for consumption is what makes EGM substantially faster per iteration than PI in this setting; without it, the two methods would coincide in cost.

Additional Comparisons. We focus the main text on discrete-state VFI and the GEE-based PI and EGM methods, since these comparisons isolate the gains from utilizing the GEE. A fourth method, VFI with Chebyshev interpolation, is described in Appendix C. That appendix also reports a separate comparison between the GEE methods and Chebyshev VFI, while Appendix C.1 reports the full grid-robustness table across all methods and grid specifications.

Implementation Details. Income follows the AR(1) process described above. For the GEE methods and discrete-state VFI, log income is discretized using the Tauchen approximation: an equally spaced log-income grid with transition probabilities obtained by integrating the normal innovation density over the grid cells.

Debt is restricted to a bounded interval $[b_{\text{min}}, b_{\text{max}}]$. The GEE methods represent inherited debt on an equally spaced grid, and EGM uses the same grid for next-period debt. Discrete-state VFI likewise uses one finite grid for inherited and next-period debt.

The GEE methods use shape-preserving cubic Hermite interpolation in the debt dimension to evaluate prices, continuation values, and their derivatives off grid. The interpolants are constructed from function values and derivative objects implied by the GEE, such as q_b and W_b.

In PI, the GEE is solved state by state by root-finding. The raw policy update is then projected, row by row in income, onto a low-order Chebyshev basis in debt; the projected policy is used in the subsequent value, price, and derivative updates, and h_b is the derivative of the same projection. In EGM, the root-finding step is replaced by the closed-form quadratic solution for consumption implied by the GEE at $\sigma = 2$ and h_b is obtained by finite differences.

The VFI methods use direct maximization over a debt-choice grid. Discrete-state VFI stores all objects directly on the finite tensor grid.

All methods evaluate the extreme-value inclusive value using the numerically stable log-sum-exp representation. The discrete-state VFI implementation also uses a small policy-inertia rule: a candidate borrowing policy replaces the previous policy only if it improves the repayment objective by more than ϵ_{pol}, removing spurious cycling caused by numerical near-ties across adjacent debt choices. Replication codes implementing the algorithms, along with the calibration routine, are available at `[repository URL upon acceptance]`.

6.3 Results

We compare the three main-text solution methods along two dimensions: numerical accuracy on a common validation grid and time to convergence. We first define accuracy using residuals of the equilibrium equations, evaluated independently of the algorithms used to compute the solutions, and then report stationary moments, speed, and robustness across grid specifications.

Accuracy Metrics. Each method m produces a candidate set of equilibrium objects

$$\widehat{\mathcal{S}}_m = \left\{ \hat{h}_m, \hat{q}_m, \hat{V}^R_m, \hat{V}^A_m, \hat{W}_m, \hat{\phi}_m \right\}.$$

We evaluate these objects on a common validation grid

$$G^{val} = Y^{val} \times B^{val},$$

whose income component Y^{val} is the common Tauchen Markov-chain grid shared by all methods, while the debt component B^{val} is finer than, and distinct from, the methods' computational debt grids. Accuracy is therefore compared in the debt dimension: between debt-grid points, each method is evaluated using its own approximation scheme— discrete VFI uses its finite state-space representation, and the GEE methods use their interpolated policy, value, and price objects.

The first validation statistic is the price-equation residual. Expectations are computed using the common discrete Markov-chain (Tauchen) approximation shared by all methods. For each $(y, b') \in G^{val}$, define

$$R^Q_q\left(y, b'\right) = \hat{q}\left(y, b'\right) - \bar{p} \sum_k \pi\left(y, y_k\right) \hat{\phi}\left(y_k, b'\right) \left[\lambda + (1-\lambda) z + (1-\lambda) \hat{q}\left(y_k, \hat{h}\left(y_k, b'\right)\right)\right].$$

The unit-free price error is

$$e^Q_q\left(y, b'\right) = \left| \frac{R^Q_q\left(y, b'\right)}{q^{rf}} \right|, \qquad q^{rf} = \frac{\bar{p}\left[\lambda + (1-\lambda) z\right]}{1 - \bar{p}(1-\lambda)}.$$

The second validation statistic is a value-consistency residual. Given the reported policy $\hat{h}$, define the policy-implied repayment value

$$\hat{V}^{R,\text{imp}}(y, b) = u\left(y - \left[\lambda + (1-\lambda) z\right] b + \hat{q}\left(y, \hat{h}(y, b)\right) \left[\hat{h}(y, b) - (1-\lambda) b\right]\right) + \beta \hat{W}\left(y, \hat{h}(y, b)\right).$$

The relative value error is

$$e_V(y,b) = \frac{\left|\hat{V}^R(y,b) - \hat{V}^{R,\text{imp}}(y,b)\right|}{\left|\hat{V}^{R,\text{imp}}(y,b)\right|}. \tag{14}$$

This residual checks whether the reported repayment value, policy, price schedule, and continuation value are mutually consistent.

For each pointwise error e, we report three summaries. The first is the supremum norm over the validation grid,

$$e_\infty = \sup_{(y,b)\in G^{val}} e(y,b).$$

The second is the uniform L^2 error,

$$e_2 = \left(\frac{1}{|G^{val}|}\sum_{(y,b)\in G^{val}} e(y,b)^2\right)^{1/2}.$$

The third is a stationary-distribution-weighted L^2 error,

$$e_{2,\mu} = \left(\sum_{(y,b)\in G^{val}} \mu(y,b)\, e(y,b)^2\right)^{1/2},$$

where $\mu(y,b) = \mu_y(y)/|G_b^{val}|$ uses the stationary distribution μ_y of the income process under the Tauchen approximation, applied uniformly across the debt dimension of the validation grid. The tables report these quantities in $\log_{10}$ units. Thus, a value of -2 corresponds to a unit-free error of 10^{-2}, while a value of -4 corresponds to an error of 10^{-4}. The supremum norm measures worst-case accuracy, whereas the two L^2 measures summarize typical errors, either uniformly over the validation grid or weighted by the states visited in the model's stationary distribution.

Benchmark Outcomes. Table 2 reports model moments under the benchmark grid configuration for the three main-text methods. The moments are computed from the stationary distribution implied by each numerical solution, using the same transformations as in Chatterjee and Eyigungor (2012). The three methods deliver very similar targeted moments: debt-to-output ratios are essentially identical, average spreads are close to the calibration target, and spread volatility and the correlation of spreads with output are of comparable magnitudes. The two GEE-based methods are nearly indistinguishable across all moments and agree closely with discrete-state VFI. The remaining discrepancies are small relative to the cross-method agreement in the targeted moments. VFI with Chebyshev interpolation

Table 2: Benchmark Outcome

Moment	Targeted	Data	Chatterjee and Eyigungor (2012)	VFI Discrete	GEE PI	GEE EGM
b/y	×	1.00	0.70	0.70	0.70	0.70
Average spread	×	0.08	0.08	0.07	0.08	0.08
$\sigma(c)/\sigma(y)$		1.09	1.11	1.09	1.09	1.09
$\sigma(\frac{y-c}{y})/\sigma(y)$		0.17	0.20	0.14	0.14	0.14
$\sigma(\text{spread})$	×	0.04	0.04	0.04	0.05	0.04
$\text{corr}(c, y)$		0.98	0.99	0.99	0.99	0.99
$\text{corr}(\frac{y-c}{y}, y)$		-0.88	-0.44	-0.59	-0.55	-0.54
$\text{corr}(\text{spread}, y)$	×	-0.79	-0.65	-0.86	-0.79	-0.81

produces very similar moments and is reported in Appendix C.

Economic Decomposition of the GEE. We now describe the behavior of the sovereign and the role of the different terms in the GEE in determining the marginal value of borrowing.

First, Figure 1 shows the decision rules of the sovereign and corresponding debt price at all points in the state space. From the bottom-left panel, we see when the sovereign enters the risky borrowing region it faces a sharp decline in the price of its debt. The borrowing policy $b' = h(y, b)$ is monotone in both y and b. The buyback level, the negative of the $b' - (1 - \lambda)b$ displayed, is not monotone, surging at high levels of debt under poor endowment realisations because of the lowered price of debt. In this specific quantitative environment, however, buyback remains negative across states—indicating continued net sales and build up of liabilities—and therefore rules out potential ridges and non-monotonicities in consumption $c(y, b)$ and borrowing.

To understand the forces that shape the optimal choices of the sovereign, we plot the left-hand side of the GEE in Figure 2. Recall from Equation (6) the derivative of the debt price is the sum of the dilution risk and default risk terms. The top-left panel shows the marginal value of borrowing —the first term on the right-hand side of Equation (12)— which is highest near the boundary of the risky-borrowing region. The next three panels decompose the marginal value of borrowing into (i) the marginal utility ignoring the derivative of the debt price $u_c(c)q(y, b')$, (ii) the marginal utility accounting for default risk $u_c(c)(b' - (1 - \lambda)b)q_b^{def}(b')$, and (iii) the marginal utility accounting for dilution risk $u_c(c)(b' - (1 - \lambda)b)q_b^{dil}(b')$, where q_b^{def} and q_b^{dil} are the default-risk and dilution-risk components of q_b identified in Equation (6). First, in the absence of the effects of the price derivative $u_c(c)q(y, b')$, the marginal utility of consumption is high when debt is high and the endowment is low, and there is incentive

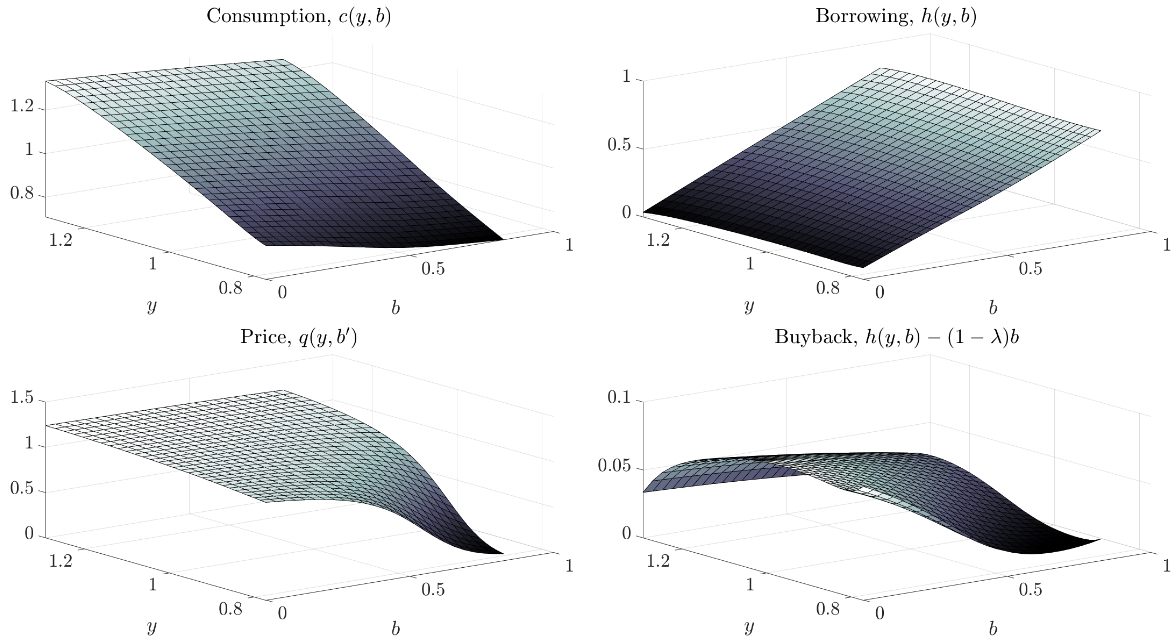

Figure 1: Policy functions evaluated at all points (y, b) in the state space.

for the sovereign to increase borrowing. Second, the effects of the price derivative in terms of default and dilution discipline the sovereign into borrowing less, which we can see in the bottom two panels, where the corresponding components of marginal revenue from additional borrowing become negative. Third, the disciplining effects of the price derivative come through the discount for both default risk and dilution risk, while the effects of dilution risk appear to be larger.

Speed, Accuracy, and Grid Robustness. Table 3 compares the GEE methods with the benchmark discrete-state VFI solution. The table reports model moments, numerical accuracy, grid sizes, and computational times. To make the accuracy comparison transparent, the benchmark VFI column reports the original $\log_{10}$ residuals, while the GEE columns report residuals relative to the corresponding VFI residual. Thus, entries below one indicate that the GEE residual is smaller than the benchmark VFI residual.

Several findings stand out. First, the model moments are stable across the methods shown in the table. The debt-to-output ratio is 0.70 in all columns, and the average spread is essentially unchanged. Consumption volatility, net-export volatility, spread volatility, and the correlations with output are also close across methods. The coarse GEE grids generate somewhat less negative net-export and spread-output correlations, but the overall stationary implications remain very similar.

Table 3: Outcomes, Accuracy, and Speed Relative to Discrete-State VFI

	VFI Discrete	GEE PI		GEE EGM	
	Benchmark	Benchmark	Coarse	Benchmark	Coarse
Stationary moments — calibration targets					
b/y	0.70	0.70	0.70	0.70	0.70
Average spread	0.079	0.080	0.080	0.080	0.080
σ(spread)	0.042	0.045	0.047	0.045	0.047
corr(spread, y)	-0.83	-0.82	-0.79	-0.82	-0.79
Additional moments					
$\sigma(c)/\sigma(y)$	1.1	1.1	1.1	1.1	1.1
$\sigma(\frac{y-c}{y})/\sigma(y)$	0.13	0.13	0.14	0.13	0.14
corr(c, y)	1.0	0.99	0.99	0.99	0.99
corr$(\frac{y-c}{y}, y)$	-0.62	-0.60	-0.54	-0.60	-0.54
Numerical accuracy: VFI Discrete column reports $\log_{10}$ *absolute residual;*					
GEE columns report the ratio of the GEE residual to the corresponding VFI residual					
Price sup	-2.75	0.0020	0.023	0.0032	0.019
Price L^2	-4.03	0.0033	0.033	0.0046	0.033
Price stat. L^2	-4.14	0.0033	0.010	0.0023	0.010
Value sup	-5.11	0.058	0.38	0.066	0.33
Value L^2	-6.32	0.14	0.90	0.15	0.91
Value stat. L^2	-6.61	0.15	0.64	0.15	0.61
Computational cost					
Grid y	101	101	101	101	101
Grid b	350	35	20	35	20
Grid b'	-	-	-	35	20
Tolerance	1e-09	1e-09	1e-09	1e-09	1e-09
Iterations to converge	549	452	455	452	452
Time to converge	132.60	102.26	96.99	11.04	10.27
Time per iteration	0.242	0.226	0.213	0.024	0.023

Note: Times are in seconds. Moments are computed from the stationary distribution implied by each numerical solution, using the same transformations as in Chatterjee and Eyigungor (2012). The benchmark discrete-state VFI column reports accuracy as $\log_{10}$ unit-free residuals on the common validation grid. The GEE accuracy entries are normalized by the corresponding benchmark VFI residual: $e_m/e_{\text{VFI}} = 10^{\log_{10}(e_m)-\log_{10}(e_{\text{VFI}})}$. Thus, values below one indicate smaller residuals than benchmark discrete-state VFI; 0.10 indicates an error one order of magnitude smaller. Price residuals are normalized by the risk-free long-term bond price, and value residuals are relative to the policy-implied repayment value. Tolerance is the sup-norm convergence criterion for the method-specific objects updated in the fixed-point iteration. For EGM, Grid b' is the exogenous next-period debt grid used to construct the endogenous current-debt grid. Times were measured on an Intel i7-10700F CPU, single-threaded MATLAB R2025b (no GPU; `parfor` disabled). All methods are implemented in pure MATLAB with vectorized inner loops; no MEX, no external solvers.

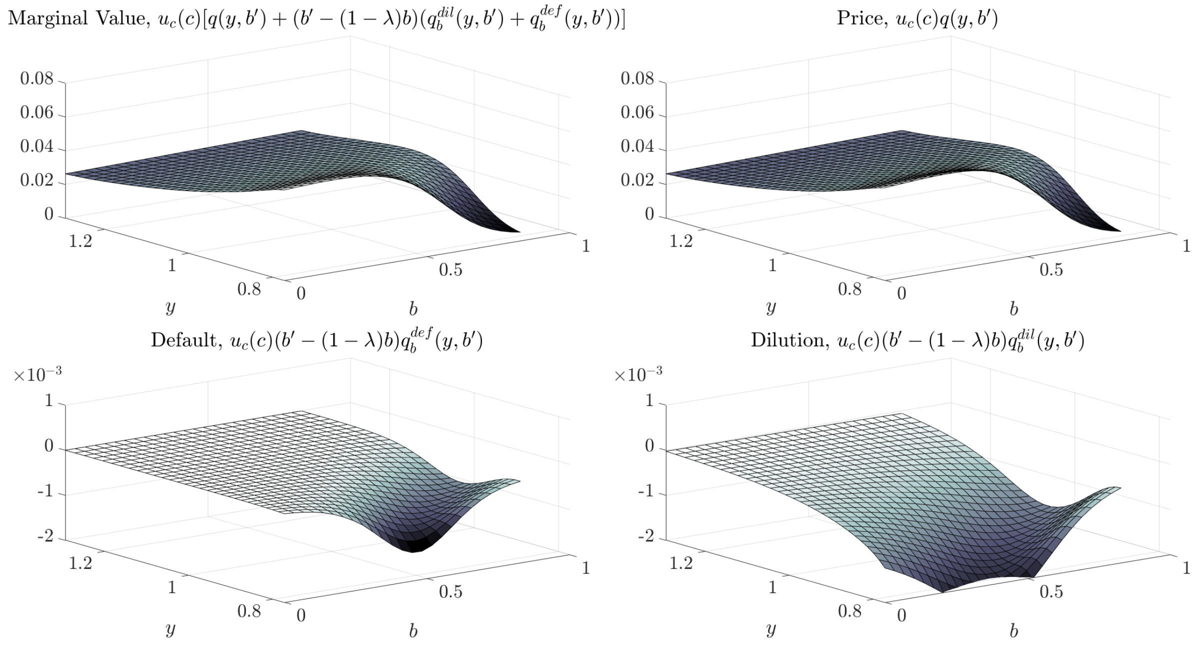

Figure 2: Decomposition of the left-hand side of the Generalized Euler Equation into its individual components evaluated at all points (y, b) in the state space.

Second, benchmark PI runs at roughly the same per-iteration cost as benchmark discrete-state VFI (0.226 versus 0.242 seconds per iteration). At this similar cost per iteration, PI delivers dramatically higher accuracy. Across the three price metrics, PI's residuals are between two and three orders of magnitude smaller than the corresponding benchmark VFI residuals (ratios of 0.0020, 0.0033, and 0.0033). PI's value residuals are also substantially smaller: ratios of 0.058 in sup norm, 0.14 in L^2, and 0.15 in stationary-weighted L^2. Because PI also converges in fewer iterations, its total time is lower than benchmark VFI: 102.26 seconds versus 132.60 seconds.

Third, coarse PI illustrates the complementary point. With a much smaller debt grid (20 versus 350 points), coarse PI is still substantially more accurate than benchmark discrete-state VFI on prices and modestly more accurate on values, while requiring less time. Coarse PI's price-residual ratios are 0.023, 0.033, and 0.010 across the three price metrics — one to two orders of magnitude smaller than under benchmark VFI. Its value-residual ratios are 0.38 in sup norm, 0.90 in L^2, and 0.64 in stationary-weighted L^2 — still below the VFI benchmark on all three, though closer than under benchmark PI. At the same time, coarse PI converges in 96.99 seconds, compared with 132.60 seconds for benchmark VFI.

Fourth, EGM shifts the speed–accuracy frontier further outward. Like PI, EGM solves the GEE system rather than directly maximizing the Bellman equation, but it avoids the state-by-state root-finding step by exploiting the closed-form quadratic for consumption available at $\sigma = 2$. This makes each iteration much cheaper: benchmark EGM requires only 0.024 seconds per iteration, compared with 0.226 seconds for benchmark PI and 0.242 seconds for benchmark discrete-state VFI. At the same time, EGM matches PI's accuracy in the reported residuals. Benchmark EGM converges in 11.04 seconds, and coarse EGM converges in 10.27 seconds, both far below the time required by benchmark discrete-state VFI.

Overall, the table isolates three margins. Holding per-iteration cost roughly fixed, benchmark PI is two to three orders of magnitude more accurate than benchmark discrete-state VFI on prices and substantially more accurate on values. Holding accuracy at VFI-level or better, coarse PI is faster than benchmark VFI. EGM then delivers the strongest result: it is both substantially faster and substantially more accurate than the VFI benchmark, reflecting the gains from solving the GEE system through an endogenous-grid construction rather than through either direct maximization or state-by-state root-finding.

The accuracy advantage is structural, not interpolation-driven. The GEE methods also dominate the smoother VFI benchmark studied in Appendix C: in Table 7, Chebyshev VFI's sup-norm value residual is about 360–370 times the VFI Discrete Benchmark residual at every grid level, while the GEE methods bring that residual to roughly 6% at the Benchmark grid and 2% at the Fine grid. The Chebyshev penalty does not shrink with grid refinement, indicating that the value-function accuracy advantage of the GEE methods reflects the GEE characterization itself rather than the choice of interpolation basis.

Convergence. All three main-text methods converge stably under the common extreme-value smoothing, reaching the prescribed sup-norm tolerance without damping. The two GEE methods have nearly identical and very regular convergence profiles, with the price and value errors declining together, while discrete-state VFI requires a longer initial adjustment phase before settling into a regular contraction-like decline (iteration counts are in Table 3). Evaluating every method under the same smoothed default decision is what isolates algorithmic differences from differences in the underlying economic problem: without continuous randomization—neither the *m*-shock of Chatterjee and Eyigungor (2012) nor extreme-value shocks—standard grid-based VFI can generate irregular policy updates and unstable convergence when the debt grid is sufficiently fine. The convergence paths of all four methods are shown in Figure 5 in Appendix C.

Accuracy: Den Haan–Marcet Test. To assess accuracy beyond moment matching and the direct residual diagnostics above, we apply the test of den Haan and Marcet (1994). The test checks whether simulated Euler-equation residuals are orthogonal to variables in the agents' information set. We simulate 2000 economies of 2000 periods each, discard the first 100 periods, and compute the test statistic using as instruments $\omega(x_t) = 1$, a χ^2_1 test, and $\omega(x_t) = [1, y_t, b_t]$, a χ^2_3 test. Table 4 reports the absolute deviation, in percentage points, of the empirical rejection rate from the nominal 5 percent level.

Relative to discrete-state VFI, the GEE methods do not merely improve the Den Haan–Marcet statistics; they effectively eliminate the systematic Euler equation rejection. Across the four reported tail/instrument combinations, the average absolute deviation from the nominal 5 percent rejection frequency is 0.19 percentage points for EGM and 0.24 percentage points for PI, compared with 3.68 percentage points for discrete-state VFI. With 2000 simulated economies, the Monte Carlo standard error of a 5 percent rejection frequency is approximately 0.49 percentage points, so the GEE deviations are indistinguishable from simulation noise. In contrast, discrete-state VFI over-rejects sharply in the upper tail: its deviations of 5.75 and 6.00 percentage points imply upper-tail rejection frequencies of about 10.75 and 11.00 percent, more than twice the nominal rate. Thus, by the Den Haan–Marcet criterion, the GEE solutions pass the Euler-equation orthogonality test, whereas the discrete-state VFI solution displays systematic Euler equation errors.

Table 4: den Haan and Marcet (1994) Accuracy Test

$\omega(x_t) = 1$			$\omega(x_t) = [1, y_t, b_t]$		
VFI	GEE		VFI	GEE	
Discrete	PI	EGM	Discrete	PI	EGM
Absolute Deviation from Lower 5% (percentage points)					
1.10	0.05	0.00	1.85	0.45	0.20
Absolute Deviation from Upper 5% (percentage points)					
5.75	0.15	0.05	6.00	0.30	0.50

Note: Absolute deviation, in percentage points, of the empirical rejection rate from the nominal 5 percent level. The statistic of den Haan and Marcet (1994) is computed from 2000 simulated economies of 2000 periods each, after discarding 100 burn-in periods. We denote by $\omega(x_t)$ the vector of instruments for the Euler-equation residuals. Small deviations indicate that the simulated Euler-equation residuals are close to orthogonal to the instruments.

Convexity with Extreme Value Shocks. The use of first-order-condition methods such as the GEE relies on the repayment objective being sufficiently well behaved in the borrowing choice. The choice set for next-period debt is convex, but the objective need not be globally concave. Nonconcavities are most likely to arise when the sovereign enters the period with high debt and receives a favorable income realization. In the limiting case with $\alpha = 0$, the default decision generates a kink in the Laffer curve near the risk-free debt limit b^*, which can imply two points at which the GEE is satisfied, one below b^* and one above it (see discussion in Section 2.3). From Proposition 2, a sufficiently large extreme-value scale parameter smooths this kink and restores the relevant convexity/concavity properties. The quantitative question is how large α must be in the calibrated model.

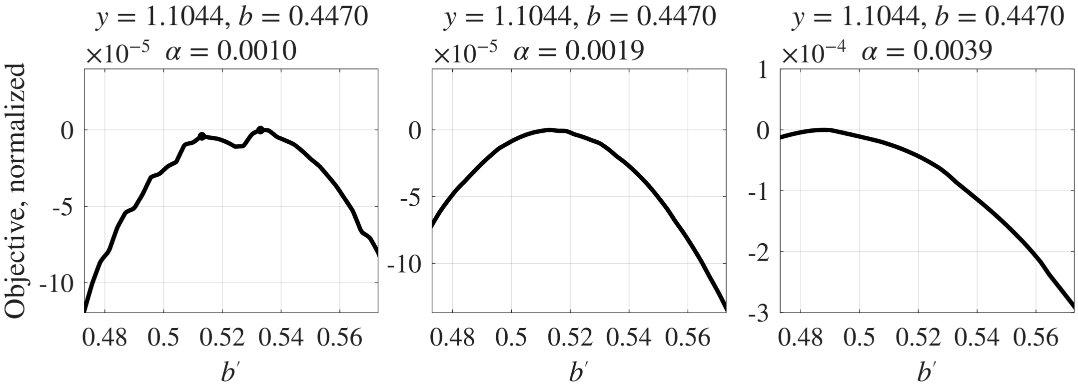

Figure 3: Repayment objective as a function of next-period debt b' for different values of the extreme-value scale parameter α. The objective is evaluated at the same state, $(y, b) = (1.1044, 0.4470)$, using the calibrated long-term debt model.

Figure 3 illustrates this mechanism for the calibrated long-term debt model. The state shown is selected because the low-α objective displays a visible nonconcavity. At $\alpha = 0.0010$ (roughly a quarter of the benchmark value) the repayment objective has two local peaks; at $\alpha = 0.0019$ the nonconcavity is largely removed; at the benchmark calibration $\alpha = 0.0039$ the objective is single peaked. Thus, in the calibrated model, the amount of extreme-value smoothing needed to eliminate economically relevant multiple local solutions is small relative to the benchmark value of α.

If α is not sufficiently large, the GEE may still exist but can have more than one solution. In that case, solving the first-order condition alone is not sufficient: one must evaluate the associated values and select the global maximum. In practice, the multiple solutions arise from the buyback region of the price schedule, so a simple safeguard is to initialize the root-finding routine from both low and high borrowing guesses (we refer to this as the multi-start GEE safeguard). If both initializations converge to the same choice, the GEE solution is unique at that state. If they converge to different choices, the algorithm evaluates the repayment objective at both candidates and selects the one with the higher value. This procedure is not needed in the benchmark computations reported above, where the calibrated value $\alpha = 0.0039$ is sufficiently smoothing for the GEE methods to converge reliably.

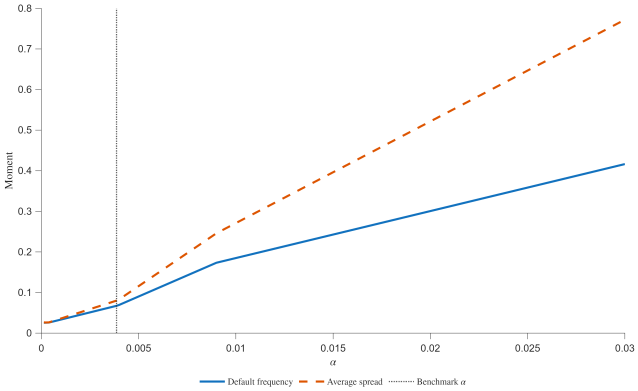

Figure 4: Default Probability and Spread for Different values of α. The calibrated benchmark value is $\alpha = 0.0039$.

Figure 4 shows how the default probability and the average spread vary with α over a wide range. The moments are stable for α values near and below our calibrated value ($\alpha = 0.0039$), confirming that our calibrated α is small enough to produce a model that is quantitatively close to the limiting case $\alpha = 0$.

On the interaction between α and the debt grid, we find no tight interaction in our calibration. Table 7 reports the four methods at three grid sizes (coarse, benchmark, and fine) for the calibrated α; the GEE methods improve monotonically with grid refinement at this α, so there is no "too-small-α-for-this-grid" breakdown of the kind one would expect if grid spacing and the buyback-region kink were interacting adversely. The practitioner-facing summary is: choose α as small as possible subject to the safeguard that the multi-start root-finding procedure described in the discussion of Figure 3 returns the same solution from low and high initializations; the grid can then be refined independently.

7 Conclusion

In this paper we have characterized the solution to the sovereign default problem with long-term debt. We have used the GEE to describe how default risk and dilution risk shape the borrowing decision. We have documented the existence of kinks in the pricing function where default risk starts, which—depending on the size of existing debt—may result in the decision being held constant for a variety of states or in a jump in debt issuance. Adding noise to the default decision, in our case in the form of extreme value shocks, eliminates these kinks and convexifies the decision problem, yielding a GEE with default and dilution risk that characterizes the solution everywhere. We have shown that an equilibrium of the finite-horizon economies exists and is unique, and that it converges as the horizon grows, establishing existence and uniqueness among the equilibria that are the limit of finite economies. We have applied our methods to a fully calibrated model following Chatterjee and Eyigungor (2012), incorporating persistent income, stochastic re-entry and output penalties upon default. Using this application, we have demonstrated the large gains in accuracy and speed obtained by policy function iteration methods that exploit the GEE. In particular, at CRRA $\sigma = 2$ the GEE for consumption admits a closed-form quadratic solution that the endogenous grid method exploits, delivering residuals orders of magnitude smaller than those of standard value function iteration at roughly an order of magnitude lower cost, with accuracy verified by the den Haan and Marcet (1994) test. The advantage extends to the smoother VFI benchmark of Appendix C: Chebyshev interpolation there does not close the value-residual gap with grid refinement, indicating that the accuracy advantage reflects the GEE characterization itself rather than the choice of basis. We have provided detailed pseudocode and implementation guidance for the discrete-state VFI, policy function iteration, and endogenous grid method algorithms. Appendix B shows that the GEE characterization itself extends to richer environments—the canonical long-term-debt model with persistent income and stochastic re-entry, the partial-default model, and a positive recovery rate upon default—so that the approach is not tied to the simplified baseline.

A scope remark. The uniqueness result of Section 4 and the GEE characterization of Section 5 refer to the *Markov* equilibrium of the long-term debt problem with exogenous fundamentals. We do not address the multiplicity of equilibria associated with rollover-driven self-fulfilling debt crises, as in Cole and Kehoe (2000) and Conesa and Kehoe (2017). Those multiplicities arise from the lender side, through self-fulfilling rollover failures at the period's debt auction, and are orthogonal to the equilibrium-selection device (strict positivity of h_t) used here. Adapting the GEE machinery to environments with rollover risk would require a separate treatment of the lender coordination problem and is left for future work.

We hope that this paper contributes to replacing black-box solution methods in the sovereign default literature with transparent, verifiable computational strategies grounded in the analytical characterization of the equilibrium.

References

Aguiar, Mark and Gita Gopinath, "Defaultable Debt, Interest Rates and the Current Account," *Journal of International Economics*, 2006, *69* (1), 64–83.

__ **and Manuel Amador**, "A contraction for sovereign debt models," *Journal of Economic Theory*, 2019, *183*, 842–875.

__ **and** __ , "Self-Fulfilling Debt Dilution: Maturity and Multiplicity in Debt Models," *American Economic Review*, September 2020, *110* (9), 2783–2818.

__ **and** __ , *The Economics of Sovereign Debt and Default*, Princeton University Press, 2021.

— , — , Hugo Hopenhayn, and Iván Werning, "Take the Short Route: Equilibrium Default and Debt Maturity," *Econometrica*, March 2019, *87* (2), 423–462.

Arellano, Cristina, "Default Risk and Income Fluctuations in Emerging Economies," *American Economic Review*, June 2008, *98* (3), 690–712.

— and Ananth Ramanarayanan, "Default and the Maturity Structure in Sovereign Bonds," *Journal of Political Economy*, April 2012, *120* (2), 187–232.

— , Lilia Maliar, Serguei Maliar, and Viktor Tsyrennikov, "Envelope Condition Method with an Application to Default Risk Models," *Journal of Economic Dynamics and Control*, August 2016, *69*, 436–459.

— , Xavier Mateos-Planas, and Jose-Victor Rios-Rull, "Partial Default," *Journal of Political Economy*, June 2023, *131* (6).

— , Yan Bai, and Gabriel P Mihalache, "Monetary Policy and Sovereign Risk in Emerging Economies (NK-Default)," Working Paper 26671, National Bureau of Economic Research January 2020.

Auclert, Adrien and Matthew Rognlie, "Unique equilibrium in the Eaton–Gersovitz model of sovereign debt," *Journal of Monetary Economics*, December 2016, *84*, 134–146.

Bhaskar, V., G. J. Mailath, and S. Morris, "A Foundation for Markov Equilibria in Sequential Games with Finite Social Memory," *Review of Economic Studies*, 2012, *80* (3), 925–948.

Bulow, J. and K. Rogoff, "Sovereign Debt: Is to Forgive to Forget?," *American Economic Review*, 1989, *79* (1), 43–50.

Carroll, Christopher D., "The method of endogenous gridpoints for solving dynamic stochastic optimization problems," *Economics Letters*, 2006, *91* (3), 312–320.

Chatterjee, S., D. Corbae, M. Nakajima, and J.-V. Ríos-Rull, "A Quantitative Theory of Unsecured Consumer Credit with Risk of Default," 2007, *75* (6), 1525–1589.

Chatterjee, Satyajit and Burcu Eyigungor, "Maturity, Indebtedness, and Default Risk," *American Economic Review*, October 2012, *102* (6), 2674–2699.

Clausen, Andrew and Carlo Strub, "Reverse Calculus and Nested Optimization," *Journal of Economic Theory*, May 2020, *187* (105019).

Cole, Harold L. and Timothy J. Kehoe, "Self-Fulfilling Debt Crises," *Review of Economic Studies*, 2000, *67* (1), 91–116.

Conesa, J. C. and T. J. Kehoe, "Gambling for Redemption and Self-Fulfilling Debt Crises," *Economic Theory*, 2017, *64* (4), 707–740.

Debortoli, Davide, Ricardo Nunes, and Pierre Yared, "Optimal Time-Consistent Government Debt Maturity," *Quarterly Journal of Economics*, February 2017, *132* (1), 55–102.

den Haan, Wouter J. and Albert Marcet, "Accuracy in Simulations," *Journal of Business & Economic Statistics*, 1994, *12* (3), 305–314.

Dvorkin, M., J.M. Sánchez, H. Sapriza, and E. Yurdagul, "Sovereign Debt Restructurings," *American Economic Journal: Macroeconomics*, 2021, *13* (2), 26–77.

Eaton, Jonathan and Mark Gersovitz, "Debt with Potential Repudiation: Theoretical and Empirical Analysis," *Review of Economic Studies*, April 1981, *48* (2), 289–309.

Faraglia, Elisa, Albert Marcet, Rigas Oikonomou, and Andrew Scott, "Government Debt Management: The Long and the Short of It," *Review of Economic Studies*, October 2019, *86* (6), 2554–2604.

Feng, Zhigang and Manuel Santos, "Sovereign Default, TFP, Fiscal and Financial Frictions," 2021. Unpublished

Manuscript, University of Miami.

Gomes, Joao, Urban Jermann, and Lukas Schmid, "Sticky Leverage," 2014. Working Paper, University of Pennsylvania.

Hatchondo, Juan Carlos and Leonardo Martinez, "Long-duration Bonds and Sovereign Defaults," *Journal of International Economics*, 2009, *79* (1), 117–125.

__ , __ , and Cesar Sosa-Padilla, "Debt Dilution and Sovereign Default Risk," *Journal of Political Economy*, 2016, *124* (5), 1383–1422.

__ , __ , and Horacio Sapriza, "Quantitative Properties of Sovereign Default Models: Solution Methods Matter," *Review of Economic dynamics*, 2010, *13* (4), 919–933.

Iskhakov, Fedor, Thomas H. Jørgensen, John Rust, and Bertel Schjerning, "The Endogenous Grid Method for Discrete-Continuous Dynamic Choice Models with (or without) Taste Shocks," *Quantitative Economics*, July 2017, *8* (2), 317–365.

Jang, Youngsoo and Soyoung Lee, "A Generalized Endogenous Grid Method for Default Risk Models," *Journal of Money, Credit and Banking*, 2024.

Kiiashko, Sergii and Lilia Maliar, "Solving Sovereign Debt Models Using Endogenous Grid Method," June 2021. Manuscript, The Graduate Center, CUNY.

Klein, Paul, Per Krusell, and José-Víctor Ríos-Rull, "Time-Consistent Public Policy," *Review of Economic Studies*, July 2008, *75* (3), 789–808.

Krusell, Per and Anthony A. Smith, "Consumption–Savings Decisions with Quasi–Geometric Discounting," *Econometrica*, January 2003, *71* (1), 365–375.

__ , Burhanettin Kuruscu, and Anthony A. Smith, "Equilibrium Welfare and Government Policy with Quasi-geometric Discounting," *Journal of Economic Theory*, 2002, *105*, 42–72.

__ , __ , and __ , "Temptation and Taxation," *Econometrica*, November 2010, *78* (6), 2063–2084.

Livshits, Igor, James MacGee, and Michelle Tertilt, "Consumer Bankruptcy: A Fresh Start," 2007, *97* (1), 402–418.

Mateos-Planas, Xavier, "Demographics and the Politics of Capital Taxation in a Life-Cycle Economy," *American Economic Review*, March 2010, *100* (1), 337–63.

Mihalache, Gabriel, "Sovereign Default Resolution through Maturity Extension," *Journal of International Economics*, 2020, *125*, 103–126.

__ , "Solving Default Models," in "Oxford Research Encyclopedia of Economics and Finance," Oxford University Press, 2025.

Niepelt, Dirk, "Debt Maturity Without Commitment," *Journal of Monetary Economics*, 2014, *68*, 37–54.

Rust, John, "Statistical Models of Discrete Choice Processes," *Transportation Research Part B: Methodological*, April 1988, *22* (2), 125–158.

Appendix

A Proofs

Proposition 1 (Vanishing Taste Shocks)**.** *As $\alpha \to 0$, the decision problem converges to the one presented in Section 2.*

Proof of Proposition 1. We begin with the repayment surplus. Consider the term $\alpha \ln\left(1 + e^{S(y',b')/\alpha}\right)$ with $S(y,b) = V^R(y,b) - V^A(y)$. For any $x \in \mathbb{R}$,

$$\lim_{\alpha \to 0} \alpha \ln\left(1 + e^{x/\alpha}\right) = \begin{cases} x & x > 0, \\ 0 & x \leq 0, \end{cases} = \max\{x, 0\}.$$

Setting $x = S(y', b')$, we obtain

$$\lim_{\alpha \to 0} \alpha \ln\left(1 + e^{S(y',b')/\alpha}\right) = \max\{S(y', b'), 0\}.$$

To be able to pass the limit through the integral, we need to apply the dominated convergence theorem. That is, for the sequence of functions $f_\alpha(y') = \alpha \ln\left(1 + e^{S(y',b')/\alpha}\right)$, we need a dominating function $g(y')$ such that $|f_\alpha(y')| \leq g(y')$ for all α sufficiently small and $\int g(y')\, dF^y(y') < \infty$. Consider two cases. First, when $S(y', b') \geq 0$:

$$\begin{aligned} \alpha \ln\left(1 + e^{S(y',b')/\alpha}\right) &\leq \alpha \ln\left(2e^{S(y',b')/\alpha}\right) \\ &= \alpha \ln(2) + S(y', b') \\ &\leq S(y', b') + C, \end{aligned}$$

where $C = \alpha_0 \ln(2)$ for some fixed small $\alpha_0 > 0$. Second, when $S(y', b') < 0$

$$\alpha \ln\left(1 + e^{S(y',b')/\alpha}\right) \leq \alpha \ln(2) \leq C.$$

Combining both cases gives

$$\left|f_\alpha(y')\right| = \alpha \ln\left(1 + e^{S(y',b')/\alpha}\right) \leq \max\{S(y', b'), 0\} + C \equiv g(y').$$

Since $\int g(y')\, dF^y(y') = \int \max\{S(y', b'), 0\}\, dF^y(y') + C < \infty$, the dominated convergence theorem applies. Therefore, passing the limit through the integral,

$$\lim_{\alpha\to 0}\int \alpha \ln\Big(1+e^{S(y',b')/\alpha}\Big)\, dF^y(y') = \int \max\{S(y',b'),0\} dF^y(y').$$

Subtracting the autarky Bellman $V^A(y) = u(y) + \beta\int V^A(y')\, dF^y(y')$ from the repayment Bellman, and using $G(y',b') - V^A(y') = \alpha\ln(1+e^{S(y',b')/\alpha}) - \alpha\ln 2$, yields a Bellman in the surplus, $S(y,b) = \max\{u(c) - u(y) + \beta\int \alpha\ln(1+e^{S/\alpha})\, dF^y - \beta\alpha\ln 2\}$. The additive constant $\beta\alpha\ln 2$ vanishes in the $\alpha\to 0$ limit. Substituting this result into the Bellman equation yields

$$S(y,b) = \max_{b'\in\Gamma(y,b;q)}\left\{u\big(y-b+q(b')\big[b'-(1-\lambda)b\big]\big) - u(y) + \beta\int \max\{S(y',b'),0\} dF^y(y')\right\}.$$

For the repayment probability, rewrite by factoring out $e^{V^R(y,b)/\alpha}$,

$$\phi(y,b) = \frac{1}{1+\exp\Big(\frac{V^A(y)-V^R(y,b)}{\alpha}\Big)}.$$

For any $x\in\mathbb{R}$,

$$\lim_{\alpha\to 0}\frac{1}{1+e^{x/\alpha}} = \begin{cases} 1 & x<0,\\ 0 & x>0.\end{cases}$$

Setting $x = V^A(y) - V^R(y,b)$, we obtain

$$\lim_{\alpha\to 0}\phi(y,b) = \begin{cases} 1 & V^R(y,b) > V^A(y),\\ 0 & V^R(y,b) < V^A(y).\end{cases}$$

At the knife-edge case in which $V^R(y,b) = V^A(y)$, the limit of the logit choice probability equals $1/2$. Since this event occurs with probability zero (by continuity of V^R, V^A in y and continuity of F^y, the level set $\{y : V^R(y,b) = V^A(y)\}$ has F^y-measure zero for each b), the exact tie-breaking rule is immaterial. We can therefore replace the limit by the indicator $\mathbb{I}_{\{V^R(b',y')\geq V^A(y')\}}$. The bond price then becomes

$$\lim_{\alpha\to 0} q(b') = \bar{p}\int \mathbb{I}_{\{V^R(b',y')\geq V^A(y')\}}\,\big[1+(1-\lambda)\, q\big(h(b',y')\big)\big]\, dF^y(y').$$

Observe that $|\phi(b',y')\,[1+(1-\lambda)q(h(b',y'))]| \leq \frac{1+r}{r+\lambda} < \infty$, using $\phi\leq 1$ and the equilibrium bond price bound $q \leq 1/(r+\lambda)$, enabling again the application of the dominated convergence theorem therefore allowing the limit to pass through the integral. See also Iskhakov et al. (2017, Theorem 5) for a related proof. $\square$

Proposition 2 (Budget Set and Objective Function)**.** *For sufficiently large α, the objective function is strongly concave and L-smooth, the marginal objective is strongly convex in b' and Lipschitz, and the budget set is convex.*

Proof of Proposition 2. With $\alpha \to \infty$, the borrower decides to default with probability $1/2$ irrespective of (y, b). The bond price for any $b' > 0$ is therefore $q(b') \equiv q = \frac{1}{1+2r+\lambda}$. Consumption is linear in (y, b, b') since $c = y - b + \frac{1}{1+2r+\lambda}[b' - (1-\lambda)b]$. As a result, the budget set is convex. In addition, observe that

$$\lim_{\alpha\to\infty} \frac{G(y,b)}{\alpha} = \lim_{\alpha\to\infty} \frac{1}{\alpha} \int_{\epsilon} V(y,b,\epsilon)\, dF_{\epsilon} = \lim_{\alpha\to\infty} \ln\left(e^{V^R(y,b)/\alpha} + e^{V^A(y)/\alpha}\right) - \ln(2) = \ln(2) - \ln(2) = 0.$$

As noted by Iskhakov et al. (2017, pp. 334-335), this implies that the objective function directly inherits the strong concavity and the L-smoothness from the properties of the instantaneous utility function when the scale parameter is sufficiently large. The convergence $G/\alpha \to 0$ is uniform on the compact state space $Y \times B$, so by continuity of the second derivatives in α the strong-concavity and L-smoothness bounds extend from the limit to a neighbourhood of $\alpha = \infty$. The same scaling argument extends to higher derivatives. Differentiating $\partial_b G = \phi(y,b)\, \partial_b V^R(y,b)$ once more,

$$\partial_b^2 G = \partial_b \phi\, \partial_b V^R + \phi\, \partial_b^2 V^R, \quad \partial_b \phi = \tfrac{1}{\alpha}\, \phi(1-\phi)\, \partial_b V^R = O(1/\alpha),$$

uniformly on $Y \times B$ since $\phi(1-\phi) \leq 1/4$. Iterating, each successive differentiation of G introduces at most one additional $1/\alpha$ factor from $\partial_b \phi$ and otherwise involves only the derivatives $\partial_b^k V^R$ ($k \leq 3$), which inherit uniform bounds from u on the compact state space by the same contraction argument as for the Hessian. Hence $\partial_b^3 G$ is uniformly bounded for α above some threshold $\overline{\alpha}$. Applying the chain rule to $\psi(y,b,b') = u(c) + \beta \int G(y',b')\, dF^y$ with $c = y - b + q(b')[b' - (1-\lambda)b]$,

$$|\partial_{b'}^3 \psi| \leq |u'''(c)|\, c_{b'}^3 + 3|u''(c)|\, |c_{b'}|\, |c_{b'b'}| + |u'(c)|\, |c_{b'b'b'}| + \beta \int |\partial_b^3 G|\, dF^y,$$

which is uniformly bounded in (y, b, b') for $\alpha \geq \overline{\alpha}$. In other words, the marginal objective $\partial_{b'}\psi$ is Lipschitz in b' with a constant uniform in (y, b) —the bound invoked in the proofs of Lemmas 1 and 2 and Proposition 6.

□

Proposition 3 (Monotonicity). *In the above environment with $T < \infty$,*

1. **Values and Decision Rules across States** *For all t,*
   - 1.1 *$V_t^R(y,b)$ is strictly increasing in y and strictly decreasing in b.*
   - 1.2 *$W_t(b)$ is non-increasing in b.*
   - 1.3 *$\chi_t(b)$ is non-increasing in b.*
   - 1.4 *If $q_t(b')$ is non-increasing in b', $h_t(y,b)$ is non-increasing in y and non-decreasing in b.*
2. **Debt Prices across States:** *For all t, $q_t(b)$ is non-increasing in b.*
3. **Values and Default over Time:** *For all $t < T$ and all (y,b),*
   - 3.1 *$S_t(y,b) \geq S_{t+1}(y,b)$.*
   - 3.2 *$Z_t(b) \geq Z_{t+1}(b)$.*
   - 3.3 *$\chi_t(b) \geq \chi_{t+1}(b)$.*

Proof of Proposition 3.1.1. Consider the feasible set given by:

$$\Gamma_t(y,b;q_t) = \left\{ b' \in B : 0 \leq y - b + q_t(b')[b' - (1-\lambda)b] \leq \overline{c} \right\}$$

Clearly, if $\hat{y} > y$ and $\tilde{b} > b$, then $\Gamma_t(y,b;q_t) \subseteq \Gamma_t(\hat{y},b;q_t)$ and $\Gamma_t(y,\tilde{b};q_t) \subseteq \Gamma_t(y,b;q_t)$. This implies the optimal choice $h = h(y,b) \in \Gamma_t(\hat{y},b;q_t)$, and $\tilde{h} = h(y,\tilde{b}) \in \Gamma_t(y,b;q_t)$. This implies monotonicity in y since,

$$\begin{aligned} V_t^R(\hat{y},b) &= u\big(\hat{y} - b + q(\hat{h})[\hat{h} - (1-\lambda)b]\big) + \beta W_{t+1}(\hat{h}) \\ &\geq u\big(\hat{y} - b + q(h)[h - (1-\lambda)b]\big) + \beta W_{t+1}(h) \\ &> u\big(y - b + q(h)[h - (1-\lambda)b]\big) + \beta W_{t+1}(h) \\ &= V_t^R(y,b), \end{aligned}$$

where the first inequality follows from the definition of optimality, and the second from the strict monotonicity of $u(c)$. This also implies monotonicity in b since,

$$\begin{aligned} V_t^R(y,b) &= u(y - b + q(h)[h - (1-\lambda)b]) + \beta W_{t+1}(h) \\ &\geq u(y - b + q(\tilde{h})[\tilde{h} - (1-\lambda)b]) + \beta W_{t+1}(\tilde{h}) \\ &> u(y - \tilde{b} + q(\tilde{h})[\tilde{h} - (1-\lambda)\tilde{b}]) + \beta W_{t+1}(\tilde{h}) \\ &= V_t^R(y,\tilde{b}), \end{aligned}$$

where the first inequality follows from the definition of optimality, and the second from the strict monotonicity of $u(c)$. ☐

Proof of Proposition 3.1.2. Consider the definition of $W_t(b)$.

$$\begin{aligned} W_t(b) &= \int\int \max\left\{V_t^R(y,b) + \epsilon^R, V_t^A(y) + \epsilon^A\right\} dF_\epsilon(\epsilon) dF^y(y), \\ &= \int \alpha \ln\left(e^{V^R(y,b)/\alpha} + e^{V^A(y)/\alpha}\right) dF^y(y) - \alpha \ln(2). \end{aligned}$$

Since $V^A(y)$ only depends on y, the proof is immediate from Proposition 3.1.1. ☐

Proof of Proposition 3.1.3. Recall the definition of the ex-ante repayment probability,

$$\chi_t(b) = \int \phi_t(y,b) dF^y(y) = \int \frac{e^{V_t^R(y,b)/\alpha}}{e^{V_t^R(y,b)/\alpha} + e^{V_t^A(y)/\alpha}} dF^y(y).$$

Since $V^A(y)$ only depends on y, the proof is immediate from Proposition 3.1.1. ☐

Proof of Proposition 3.1.4. Similar to Chatterjee and Eyigungor (2012), define the loss function of choosing b_0' instead of b_1' in state (y,b) by

$$\Delta(b_0', b_1'|b) = (1-\lambda)b_0\left[q_t(b_1') - q_t(b_0')\right] + q_t(b_0')b_0' - q_t(b_1')b_1'.$$

Observe that Δ does not depend on y given that y is i.i.d. distributed.

For the first part of the proposition, fix y and consider two debt levels b_0 and b_1 such that $b_1 > b_0$. Assume that in b_0, the borrower optimally chooses b_0' and obtains a consumption level c_0. This means that for a $\hat{b}' < b_0'$ leading to a consumption level $\hat{c}$, we have by optimality that

$$u(c_0) + \beta W_{t+1}(b_0') \geq u(\hat{c}) + \beta W_{t+1}(\hat{b}'). \tag{15}$$

As $W_{t+1}(b_0') \leq W_{t+1}(\hat{b}')$ from Proposition 3.1.2, it must be that $c_0 \geq \hat{c}$. Now observe that $\Delta(b_0', \hat{b}'|b_0) = c_0 - \hat{c} \geq 0$ and

$$\Delta(b_0', \hat{b}'|b_0) - \Delta(b_0', \hat{b}'|b_1) = (1-\lambda)(b_0 - b_1)\big[q_t(\hat{b}') - q_t(b_0')\big] \leq 0,$$

where the inequality comes form the fact that $b_1 > b_0$ and that $q_t(b')$ is non decreasing in b'. This means that the loss in b_1 is at least as large as the loss in b_0. With this define $\tilde{c}$ being the consumption level in state b_1 choosing b_0'. By the budget constraint, it directly follows that $\tilde{c} < c_0$ given that $b_1 > b_0$. Combining this with the strong concavity of $u(\cdot)$, we get

$$\begin{aligned} u(\tilde{c}) - u(\tilde{c} - \Delta(b_0', \hat{b}'|b_1)) &> u(c_0) - u(c_0 - \Delta(b_0', \hat{b}'|b_1)) \\ &\geq u(c_0) - u(c_0 - \Delta(b_0', \hat{b}'|b_0)) \\ &= u(c_0) - u(\hat{c}) \geq 0, \end{aligned}$$

where the first inequality comes from $c_0 > \tilde{c}$, the second from $\Delta(b_0', \hat{b}'|b_1) \geq \Delta(b_0', \hat{b}'|b_0)$ and the third from the definition of $\Delta(b_0', \hat{b}'|b_0)$. This means that the wedge in utility between $\tilde{c}$ and $\tilde{c} - \Delta(b_0', \hat{b}'|b_1)$ is larger than the wedge in utility between c_0 and $\hat{c} = c_0 - \Delta(b_0', \hat{b}'|b_0)$. By (15), this implies that

$$u(\tilde{c}) + \beta W_{t+1}(b_0') > u(\tilde{c} - \Delta(b_0', \hat{b}'|b_1)) + \beta W_{t+1}(\hat{b}').$$

Hence it cannot be that in b_1, the optimal choice b' is lower than b_0'. The bond policy function is therefore non decreasing in b.

For the second part of the proposition, fix b and consider two income levels such that $0 < y_1 < y_0$. As before, assume that in y_0, the borrower optimally chooses b_0' and obtains a consumption level c_0. Considering a $\hat{b}' < b_0'$ leading to a consumption level $\hat{c}$, we get the same argument around (15) as before. With this define $\tilde{c}$ being the consumption level in state y_1 choosing b_0'. From the budget constraint, $\tilde{c} = c_0 + y_1 - y_0 < c_0$ as $0 < y_1 < y_0$. Combining this with the strong concavity of $u(\cdot)$, we get

$$u(\tilde{c}) - u(\tilde{c} - \Delta(b_0', \hat{b}'|b)) > u(c_0) - u(c_0 - \Delta(b_0', \hat{b}'|b)) = u(c_0) - u(\hat{c}) \geq 0.$$

The wedge in utility between $\tilde{c}$ and $\tilde{c} - \Delta(b'_0, \hat{b}'|b)$ is larger than the wedge in utility between c_0 and $\hat{c} = c_0 - \Delta(b'_0, \hat{b}'|b)$. By (15), it cannot be that in y_1, the optimal choice b' is lower than b'_0. The bond policy function is therefore non increasing in y. ☐

Proof of Proposition 3.2. We prove this statement by taking the limit of finite horizon. The bond price is given by

$$q_t(b) = \overline{p}\ \chi_{t+1}(b) + \overline{p}(1-\lambda)\int_{\underline{y}}^{\overline{y}} \phi_{t+1}(y,b)\ q_{t+1}\big(h_{t+1}(y,b)\big)\ dF^y.$$

The proof goes by backward induction. In the last period T, under Assumption 2, the borrower issues zero debt, i.e. $b_{T+1} = 0$, and the default decision depends on the exact realization of (ϵ^R, ϵ^A). The bond price in $T-1$ is given by

$$q_{T-1}(b) = \frac{2}{1+2r+\lambda}\chi_T(b).$$

From Proposition 3.1.3, for any $0 < b_1 < b_2$,

$$q_{T-1}(b_1) - q_{T-1}(b_2) = \frac{2}{1+2r+\lambda}\left[\chi_T(b_1) - \chi_T(b_2)\right] \geq 0.$$

Subsequently, in period $T-2$, we have that

$$\begin{aligned}
q_{T-2}(b_1) - q_{T-2}(b_2) =& \bar{p}\left[\chi_{T-1}(b_1) - \chi_{T-1}(b_2)\right] \\
& + \bar{p}(1-\lambda)\int_{\underline{y}}^{\bar{y}} \phi_{T-1}(y,b_1)q_{T-1}(h_{T-1}(y,b_1)) - \phi_{T-1}(y,b_2)q_{T-1}(h_{T-1}(y,b_2))dF^y \\
\geq& \bar{p}\left[\chi_{T-1}(b_1) - \chi_{T-1}(b_2)\right] \\
& \underbrace{+ \bar{p}(1-\lambda)\int_{\underline{y}}^{\bar{y}} \phi_{T-1}(y,b_2)\left\{q_{T-1}(h_{T-1}(y,b_1)) - q_{T-1}(h_{T-1}(y,b_2))\right\}dF^y}_{\equiv A},
\end{aligned}$$

where the inequality comes from Proposition 3.1.3. Relying on the same proposition we have that $\chi_{T-1}(b_1) - \chi_{T-1}(b_2) \geq 0$. In addition, from Proposition 3.1.4, $h_{T-1}(y,b_1) \leq h_{T-1}(y,b_2)$ implying that $A \geq 0$ given that $q_{T-1}(b_1) - q_{T-1}(b_2) \geq 0$ for $b_1 < b_2$. Hence,

$$q_{T-2}(b_1) - q_{T-2}(b_2) \geq 0.$$

Repeating the argument until $t = 0$ completes the proof. $\square$

Proof of Proposition 3.3.1. Observe that

$$V_t^A(y) = u(y) + \sum_{j=t+1}^{T} \beta^{j-t} \int_{\underline{y}}^{\overline{y}} u(y')dF^y(y'),$$
$$V_{t+1}^A(y) = u(y) + \sum_{j=t+2}^{T} \beta^{j-t-1} \int_{\underline{y}}^{\overline{y}} u(y')dF^y(y').$$

Hence

$$V_t^A(y) - V_{t+1}^A(y) = \beta^{T-t} \int_{\underline{y}}^{\overline{y}} u(y')dF^y(y').$$

Now consider the following replication strategy. Suppose that at time t, the borrower chooses to mimic the optimal borrowing policy of the subsequent period h_{t+1}, from time t to $T-1$. In that logic, at time $T-1$, the borrower issues no new debt and consumes in T its endowment y_T.

The borrower following the replication strategy in t has the same consumption as the optimizing borrower in $t+1$. To see this, denote the price and policies of the borrower following the replication strategy with $\tilde{q}_t$, $\tilde{c}_t$, $\tilde{\phi}_t$ and $\tilde{\chi}_t$. Since the borrower following the replication strategy does not issue new debt in $T-1$, it holds that $\tilde{c}_{T-1}(y,b) = c_T(y,b) = y - b[1 + (1-\lambda)\frac{1}{1+2r+\lambda}]$ and

$$\tilde{q}_{T-2}(b') = \bar{p} \int_{\underline{y}}^{\bar{y}} \tilde{\phi}_{T-1}(y',b')[1 + (1-\lambda)\tilde{q}_{T-1}(0)]dF^y(y') = \tilde{\chi}_{T-1}(b')\frac{2}{1+2r+\lambda},$$

where we used the fact that $\tilde{q}_{T-1}(0) = \frac{1}{1+2r+\lambda}$. The latter follows from $\chi_T(0) = 1/2$ and the pricing formula: $\tilde{q}_{T-1}(0) = \bar{p} \cdot \frac{1}{2} \cdot [1 + (1-\lambda)\frac{1}{1+2r+\lambda}] = \frac{1}{1+2r+\lambda}$, which is precisely the terminal price in Assumption 2. This self-consistency is the key reason for the choice of terminal price. Observe that

$$\begin{aligned}\tilde{\phi}_{T-1}(y,b) &= \frac{e^{[u(y-b[1+(1-\lambda)\frac{1}{1+2r+\lambda}]) + \beta\int u(y')dF^y(y')]/\alpha}}{e^{[u(y-b[1+(1-\lambda)\frac{1}{1+2r+\lambda}]) + \beta\int u(y')dF^y(y')]/\alpha} + e^{[u(y)+\beta\int u(y')dF^y(y')]/\alpha}} \\ &= \frac{e^{u(y-b[1+(1-\lambda)\frac{1}{1+2r+\lambda}])/\alpha}}{e^{u(y-b[1+(1-\lambda)\frac{1}{1+2r+\lambda}])/\alpha} + e^{u(y)/\alpha}} = \phi_T(y,b).\end{aligned}$$

This implies that $\tilde{q}_{T-2}(b') = \tilde{\chi}_{T-1}(b')\frac{2}{1+2r+\lambda} = \chi_T(b')\frac{2}{1+2r+\lambda} = q_{T-1}(b')$ for all b' and

$$\tilde{c}_{T-2}(y,b) = y - b + \tilde{q}_{T-2}(h_{T-1}(y,b))(h_{T-1}(y,b) - (1-\lambda)b)$$

$$
\begin{aligned}
&= y - b + q_{T-1}(h_{T-1}(y,b))(h_{T-1}(y,b) - (1-\lambda)b) \\
&= c_{T-1}(y,b).
\end{aligned}
$$

Using the same argument as before, one can then show that $\tilde{\phi}_{T-2}(y,b) = \phi_{T-1}(y,b)$ and that

$$
\begin{aligned}
\tilde{q}_{T-3}(b') &= \bar{p} \int_{\underline{y}}^{\bar{y}} \tilde{\phi}_{T-2}(y',b')[1 + (1-\lambda)\tilde{q}_{T-2}(h_{T-1}(y',b'))] dF^y(y') \\
&= \bar{p} \int_{\underline{y}}^{\bar{y}} \phi_{T-1}(y',b')[1 + (1-\lambda)q_{T-1}(h_{T-1}(y',b'))] dF^y(y') \\
&= q_{T-2}(b').
\end{aligned}
$$

This then leads to $\tilde{c}_{T-3}(y,b) = c_{T-2}(y,b)$. Repeating this argument for all remaining periods shows the consumption from $t+1$ to T of the borrower entering state (y,b) at time t, which follows the replication strategy, is equal to the consumption of an optimizing borrower from $t+1$ to T entering the same state (y,b) at time $t+1$.

In addition, the borrower at time t also has an additional period of consumption y_T. As this strategy is sub-optimal, we have that

$$
\begin{aligned}
V_t^R(y,b) \geq & u(c_{t+1}(y,b)) + \mathbb{E}_t \Bigg\{ \sum_{j=t+1}^{T-1} \beta^{j-t} \max\left\{u(c_{j+1}(y_j,b_j)) + \epsilon^R, u(y_j) + \epsilon^A\right\} \\
& + \beta^{T-t} \max\left\{u(y_T) + \epsilon^R, u(y_T) + \epsilon^A\right\} \Bigg\} \\
= & V_{t+1}^R(y,b) + \mathbb{E}_t \beta^{T-t} u(y_T).
\end{aligned}
$$

From the previous equation, we obtain that

$$
V_t^R(y,b) - V_{t+1}^R(y,b) \geq \mathbb{E}_t \beta^{T-t} u(y_T) = V_t^A(y) - V_{t+1}^A(y).
$$

This shows the monotonicity of the repayment surplus. □

Proof of Proposition 3.3.2. Consider the definition of $Z_t(b)$.

$$
Z_t(b) = \int_{\underline{y}}^{\bar{y}} \int_{\epsilon} \max\left\{S_t(y,b) + \epsilon^R - \epsilon^A, 0\right\} \, dF_\epsilon dF^y = \int_{\underline{y}}^{\bar{y}} \alpha \ln\left(1 + e^{S_t(y,b)/\alpha}\right) \, dF^y.
$$

Since $S_t(y,b) = V_t^R(y,b) - V_t^A(y)$, the proof is immediate from Proposition 3.3.1. □

Proof of Proposition 3.3.3. Recall the definition of the repayment probability,

$$\phi_t(y,b) = \frac{e^{V_t^R(y,b)/\alpha}}{e^{V_t^R(y,b)/\alpha} + e^{V_t^A(y)/\alpha}} = \frac{e^{S_t(y,b)/\alpha}}{1 + e^{S_t(y,b)/\alpha}}.$$

Observe that

$$\frac{\partial \phi_t(y,b)}{\partial S_t(y,b)} = \frac{1}{\alpha} \frac{e^{S_t(y,b)/\alpha}}{(1 + e^{S_t(y,b)/\alpha})^2} > 0.$$

Hence, it suffices to show that

$$S_t(y,b) = V_t^R(y,b) - V_t^A(y) \geq V_{t+1}^R(y,b) - V_{t+1}^A(y) = S_{t+1}(y,b),$$

which follows from Proposition 3.3.1. Furthermore, recall that $\chi_t(b) = \int_y \phi_t(y,b)\, dF^y$. Hence, the monotonicity of $\phi_t(y,b)$ in t implies the monotonicity of $\chi_t(b)$ in t. □

Lemma 1 (Uniform Boundedness)**.** *For a sufficiently large α, the sequences $\{\psi_t\}$, $\{S_t\}$, $\{h_t\}$ and $\{q_t\}$ are uniformly bounded. Also, for all t and $b' \in B$, there exist constants $\bar{q}' \geq 0$, $\bar{q}'' \geq 0$, $M \geq 0$ and $P \geq 0$ such that*

$$\sup_{t,b'} \left| \frac{\partial q_t(b')}{\partial b'} \right| \leq \bar{q}', \ \sup_{t,b'} \left| \frac{\partial^2 q_t(b')}{\partial b'^2} \right| \leq \bar{q}'', \ \sup_{t,y,b,b'} \|\nabla \partial_{b'} \psi_t(y,b,b')\| \leq M, \ \sup_{t,y,b,b'} \|\nabla^2 \partial_{b'} \psi_t(y,b,b')\| \leq P.$$

Proof of Lemma 1. We start with the bond price and the bond policy. Since $q_t \in [0, \frac{1}{r+\lambda}]$ for all t, the bond price is uniformly bounded. For the bond policy, uniform boundedness comes from the fact that $h_t \in B$ for all t. For the sequences $\{\psi_t\}$ and $\{S_t\}$, define the objective function for $t \leq T$ as

$$\psi_t(y,b,b') = u(y - b + q_t(b')[b' - (1-\lambda)b]) - u(y) + \beta \int \alpha \ln \left(1 + e^{S_{t+1}(y',b')/\alpha}\right) dF^y(y'),$$

as well as $c_t(y,b,b') = y - b + q_t(b')[b' - (1-\lambda)b]$, $A \equiv \sup_{t,y,b,b'}\{u(c_t(y,b,b')) - u(y)\}$ and $D \equiv \inf_{t,y,b,b'}\{u(c_t(y,b,b')) - u(y)\}$. The set of all triples (y,b,b') is bounded, so continuity of $u(\cdot)$ implies the finite constants A and D exist and do not depend on t. Thus for every feasible triple and every t,

$$D \leq u(c_t(y,b,b')) - u(y) \leq A. \tag{16}$$

Regarding the continuation value, for any $x \in \mathbb{R}$ and $\alpha > 0$,

$$\alpha \ln \left(1 + e^{x/\alpha}\right) \le \max\{0, x\} + \alpha \ln 2.$$

Indeed, if $x/\alpha \le 0$ then $\ln(1 + e^{x/\alpha}) \le \ln 2$; if $x/\alpha > 0$ then $\ln(1 + e^{x/\alpha}) = x/\alpha + \ln(1 + e^{-x/\alpha}) \le x/\alpha + \ln 2$. Let $K_t \equiv \sup_{y,b} S_t(y, b)$. Using (16), the upper bound is

$$\begin{aligned} S_t(y, b) &\le A + \beta \int \big(\max\{0, S_{t+1}(y', b')\} + \alpha \ln 2\big)\, dF^y(y') \\ &\le A + \beta\big(\max\{0, K_t\} + \alpha \ln 2\big). \end{aligned}$$

Taking the supremum over (y, b) yields $K_t \le A + \beta\alpha \ln 2 + \beta \max\{0, K_{t+1}\}$. Then set $C \equiv A + \beta\alpha \ln 2$. We then get $K_t \le C + \beta \max\{0, K_{t+1}\}$. If $K_{t+1} \le 0$ then $K_t \le C$. If $K_{t+1} > 0$ then $K_t \le C + \beta K_{t+1}$. Since $K_T = A$, for all t,

$$\sup_t K_t \le C/(1 - \beta) < \infty$$

For the lower bound, let $k_{t+1} \equiv \inf_{y,b} S_{t+1}(y, b)$. From (16) and $\alpha \ln(1+e^{x/\alpha}) \ge 0$, we have $S_t(y, b) \ge D + \beta \int 0\, dF^y = D$ so $k_t \ge D$ for all t. Hence, for all t, y, b,

$$D \le S_t(y, b) \le C/(1 - \beta).$$

Thus $\{S_t\}$ is uniformly bounded and so is $\{\psi_t\}$ by definition of the objective function. Turning to the derivatives, observe that

$$\begin{aligned} \frac{\partial \psi_t(y, b, b')}{\partial b'} =& u_c(c_t(y, b))[q_t(b') + q_{t,b}(b')(b' - (1 - \lambda)b)] + \beta \int \frac{e^{S_{t+1}(y',b')/\alpha}}{1 + e^{S_{t+1}(y',b')/\alpha}} \frac{\partial S_{t+1}(y', b')}{\partial b'} dF^y, \\ \frac{\partial^2 \psi_t(y, b, b')}{\partial b'^2} =& u_{cc}(c_t(y, b))[q_t(b') + q_{t,b}(b')(b' - (1 - \lambda)b)]^2 \\ &+ u_c(c_t(y, b))[2q_{t,b}(b') + q_{t,bb}(b')(b' - (1 - \lambda)b)] \\ &+ \beta \int \left[\frac{e^{S_{t+1}(y',b')/\alpha}}{(1 + e^{S_{t+1}(y',b')/\alpha})^2} \Big(\frac{\partial S_{t+1}(y', b')}{\partial b'} \Big)^2 \frac{1}{\alpha} + \frac{e^{S_{t+1}(y',b')/\alpha}}{1 + e^{S_{t+1}(y',b')/\alpha}} \frac{\partial^2 S_{t+1}(y', b')}{\partial b'^2} \right] dF^y, \\ \frac{\partial^2 \psi_t(y, b, b')}{\partial b \partial b'} =& - u_{cc}(c_t(y, b))[q_t(b') + q_{t,b}(b')(b' - (1 - \lambda)b)][1 + (1 - \lambda)q_t(b')] \\ &- u_c(c_t(y, b))q_{t,b}(b')(1 - \lambda), \end{aligned}$$

$$\frac{\partial^2 \psi_t(y,b,b')}{\partial y \partial b'} = u_{cc}(c_t(y,b))[q_t(b') + q_{t,b}(b')(b' - (1-\lambda)b)].$$

Given that the objective function is strongly concave and L-smooth under sufficiently large α, there exists a $L > 0$ such that $|\frac{\partial^2 \psi_t}{\partial b'^2}| \leq L$ for all t. As $b' \in [\underline{b}, \overline{b}]$ lies in a compact set, the gradient is Lipschitz and bounded as well. Hence the sequence $\{\partial_{b'}\psi_t\}$ is uniformly bounded. Given this, we can show that $q_{t,b}(b')$ is uniformly bounded as all the other terms in the gradient are uniformly bounded. We already noted that S_{t+1} is bounded for all t. Additionally, we have $\frac{\partial \psi_t(y,b,b')}{\partial b} = -u_c(c_t(y,b))[1 + (1-\lambda)q_t(b')]$ and by the envelope theorem $\frac{\partial \psi_t(y,b,b')}{\partial b}|_{b'=h_t(y,b)} = \frac{\partial S_t(y,b)}{\partial b}$. Given this, since $c_t(y,b)$ and $q_t(b)$ are bounded so is $\frac{\partial S_t(y,b)}{\partial b}$ for all t. Hence, $q_{t,b}(b')$ is bounded and that for all t. Thus, there exists a $\bar{q}'$ such that $\sup_{t,b'} \left|\frac{\partial q_t(b')}{\partial b'}\right| \leq \bar{q}'$.

For the second derivatives, we know that $|\frac{\partial^2 \psi_t}{\partial b'^2}| \leq L$ for all t. For the terms appearing in $\frac{\partial^2 \psi_t}{\partial b'^2}$, we already said that c_t, q_t, S_{t+1}, $\frac{\partial S_t(y,b)}{\partial b}$ and $q_{t,b}$ are bounded. Moreover, $\partial^2_{b'b'} S_t$ is uniformly bounded due to strong concavity and L-smoothness of the objective function. The only remaining term is $q_{t,bb}$ which must be bounded given that all other terms are. Hence, there exists a $\bar{q}''$ such that $\sup_{t,b'} \left|\frac{\partial^2 q_t(b')}{\partial b'^2}\right| \leq \bar{q}''$.

Since $\frac{\partial^2 \psi_t(y,b,b')}{\partial b \partial b'}$ and $\frac{\partial^2 \psi_t(y,b,b')}{\partial y \partial b'}$ combine the same aforementioned terms, the cross partial derivatives are bounded as well. As a result, the sequence $\{\nabla \partial_{b'} \psi_t(y,b,b')\} = \{(\frac{\partial^2 \psi_t}{\partial y \partial b'}, \frac{\partial^2 \psi_t}{\partial b \partial b'}, \frac{\partial^2 \psi_t}{\partial b'^2})\}$ is uniformly bounded. Hence, there exists $M \geq 0$ such that for all t, $\|\nabla \partial_{b'} \psi_t(y,b,b')\| \leq M$.

For the third derivatives of the objective function, we formulate a similar argument. From Proposition 2, the gradient inherits strong convexity and L-smoothness from the marginal utility function meaning that there exists a $L > 0$ such that $|\frac{\partial^3 \psi_t}{\partial b'^3}| \leq L$ for all t. Observe that

$$\begin{aligned}
\frac{\partial^3 \psi_t(y,b,b')}{\partial b'^3} =& u_{ccc}(c_t(y,b))[q_t(b') + q_{t,b}(b')(b' - (1-\lambda)b)]^3 \\
&+ u_c(c_t(y,b))[3q_{t,bb}(b') + q_{t,bbb}(b')(b' - (1-\lambda)b)] \\
&+ 3u_{cc}(c_t(y,b))[q_t(b') + q_{t,b}(b')(b' - (1-\lambda)b)][2q_{t,b}(b') + q_{t,bb}(b')(b' - (1-\lambda)b)] \\
&+ \beta \int \Bigg[\frac{e^{S_{t+1}(y',b')/\alpha}(1 - e^{S_{t+1}(y',b')/\alpha})}{(1 + e^{S_{t+1}(y',b')/\alpha})^3} \Big(\frac{\partial S_{t+1}(y',b')}{\partial b'} \Big)^3 \Big(\frac{1}{\alpha} \Big)^2 \\
&\qquad + \frac{e^{S_{t+1}(y',b')/\alpha}}{1 + e^{S_{t+1}(y',b')/\alpha}} \frac{\partial^3 S_{t+1}(y',b')}{\partial b'^3} \\
&+ 3 \frac{e^{S_{t+1}(y',b')/\alpha}}{(1 + e^{S_{t+1}(y',b')/\alpha})^2} \frac{\partial S_{t+1}(y',b')}{\partial b'} \frac{\partial^2 S_{t+1}(y',b')}{\partial b'^2} \frac{1}{\alpha} \Bigg] dF^y.
\end{aligned}$$

The only terms we have not yet shown are bounded are $\partial^3_{b'b'b'} S_{t+1}$ and $q_{t,bbb}$. The former is bounded by the fact that the gradient is strongly convex and L-smooth. The latter being the only term left must be bounded since all the other terms are bounded and $|\frac{\partial^3 \psi_t}{\partial b'^3}| \leq L$. Since there are no other term appearing in $\nabla^2 \partial_{b'} \psi_t(y, b, b')$, there exists $P \geq 0$ such that for all t, $\|\nabla^2 \partial_{b'} \psi_t(y, b, b')\| \leq P$. □

Lemma 2 (Equicontinuity)**.** *For a sufficiently large α, the sequences $\{S_t\}$ and $\{h_t\}$ are equicontinuous on B, the sequence $\{q_t\}$ is equicontinuous on B and the sequence $\{\psi_t\}$ is equicontinuous on $Y \times B \times B$.*

Proof of Lemma 2. The feasible set Γ is nonempty, compact-valued, and continuous. Define the objective function for $t \leq T$ as

$$\psi_t(y, b, b') = u(y - b + q_t(b')[b' - (1 - \lambda)b]) - u(y) + \beta \int \alpha \ln \left(1 + e^{S_{t+1}(y', b')/\alpha}\right) dF^y(y').$$

Fix $\varepsilon > 0$ and, for each (y, b), pick $\tilde{b}_{(y,b)} \in \Gamma(y, b, q_t)$ such that $S_t(y, b) \leq \psi_t(y, b, \tilde{b}_{(y,b)}) + \varepsilon/3$. Since Γ is continuous and compact-valued in (y, b), there exists $\delta_{(y,b)} > 0$ such that for all (y', b') with $\|(y', b') - (y, b)\| < \delta_{(y,b)}$, we can select $\tilde{b}_{(y',b')} \in \Gamma(y', b', q_t)$ satisfying $\tilde{b}_{(y',b')} \to \tilde{b}_{(y,b)}$ as $(y', b') \to (y, b)$. By joint continuity of ψ_t following from the continuity of $u(\cdot)$ and the application of the dominated convergence theorem, there exists $\delta_{(y,b)} > 0$ such that for all $(y', b') \in B_{\delta_{(y,b)}}(y, b)$,

$$|\psi_t(y, b, \tilde{b}_{(y,b)}) - \psi_t(y', b', \tilde{b}_{(y',b')})| < \varepsilon/3.$$

This implies that $\psi_t(y, b, \tilde{b}_{(y,b)}) < \psi_t(y', b', \tilde{b}_{(y',b')}) + \varepsilon/3$. Therefore, for such (y', b'),

$$\begin{aligned} S_t(y, b) &\leq \psi_t(y, b, \tilde{b}_{(y,b)}) + \varepsilon/3 \\ &\leq \psi_t(y', b', \tilde{b}_{(y',b')}) + 2\varepsilon/3 \\ &\leq S_t(y', b') + 2\varepsilon/3. \end{aligned}$$

A symmetric argument shows $S_t(y', b') \leq S_t(y, b) + 2\varepsilon/3$. Hence,

$$|S_t(y, b) - S_t(y', b')| < \varepsilon.$$

This shows that for every (y, b), there exists $\delta_{(y,b)} > 0$ such that all S_t satisfy the equicontinuity condition in the ball $B_{\delta_{(y,b)}}(y, b)$. By compactness of the domain, a finite subcover of such neighborhoods yields a uniform $\delta > 0$ such that for all $(y, b), (y', b')$ with $\|(y', b') - (y, b)\| < \delta$ and all t, $|S_t(y, b) - S_t(y', b')| < \varepsilon$. Therefore, the sequence $\{S_t\}$ is equicontinuous on $Y \times [0, \bar{b}]$.

Now consider the bond price. Define $\delta \equiv \frac{\varepsilon}{\bar{q}'}$ with $\varepsilon > 0$. Following Lemma 1, for all t and all $(b_1', b_2') > 0$ such that $|b_1' - b_2'| < \delta$, the mean value theorem implies that there exists $\xi \in (b_1', b_2')$ such that

$$|q_t(b_1') - q_t(b_2')| = \left|\frac{\partial q_t(\xi)}{\partial b'}\right| \times |b_1' - b_2'| \leq \bar{q}'|b_1' - b_2'| < \bar{q}' \times \frac{\varepsilon}{\bar{q}'} = \varepsilon.$$

Hence, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|b_1' - b_2'| < \delta$ with $|q_t(b_1', y) - q_t(b_2', y)| < \varepsilon$ uniformly over t. Therefore, the sequence $\{q_t\}$ is equicontinuous on B.

For the bond policy, let $\varepsilon > 0$. Given the equicontinuity of $q_t(b')$ on B and the compactness of Γ, $c_t(y, b, b')$ is equicontinuous in (y, b, b') for $b \in B, b' \in B$ and so is the composition $u \circ c$. Given this, the equicontinuity of $S_t(y', b')$ combined with the fact that the softmax function is Lipschitz in S_t on bounded intervals imply that ψ_t is equicontinous in $Y \times B \times B$. Fix t, and let (y, b) be arbitrary. Since the maximizer is unique and the function ψ_t is continuous, there exists $\eta > 0$ such that

$$\psi_t(y, b, h_t(y, b)) \geq \psi_t(y, b, b') + 2\varepsilon \quad \text{for all } b' \notin B_\eta(h_t(y, b)) \text{ and } b' > 0,$$

where $B_\eta(h_t(y, b))$ is the open ball of radius η around $h_t(y, b)$. Fix any $\varepsilon > 0$. By equicontinuity of ψ_t, there exists $\delta > 0$ such that if $\|(y', b') - (y, b)\| < \delta$, then

$$\sup_{b'' \in \Gamma(y,b,q_t) \cup \Gamma(y',b',q_t)} \left|\psi_t(y', b', b'') - \psi_t(y, b, b'')\right| < \varepsilon,$$

implying the ranking of maximizers is preserved for nearby inputs. As before, compactness and continuity ensure one can combine both feasible sets. This leads to $\psi_t(y', b', b'') \leq \psi_t(y, b, b'') + \varepsilon \leq \psi_t(y, b, h_t(y, b)) - \varepsilon \leq \psi_t(y', b', h_t(y, b))$ for all $b'' \notin B_\eta(h_t(y, b))$. Hence, the maximizer in (y', b') cannot occur outside $B_\eta(h_t(y, b))$ and therefore $h_t(y', b') \in B_\eta(h_t(y, b))$. As a result, $\|h_t(y', b') - h_t(y, b)\| < \eta$ showing that h_t is equicontinuous on $Y \times B$. Since the argument holds uniformly over t, the sequence $\{h_t\}$ is uniformly equicontinuous on $Y \times B$. □

Proposition 4 (Existence and Uniqueness). *Assume α is sufficiently large and $S_t(y, b) \geq S_{t+1}(y, b)$ for all $t < T$ and let $\{S_t, h_t, q_t\}_{t=0}^{\infty}$ denote the sequence generated by the recursive system:*

$$S_t(y, b) = \max_{b' \in \Gamma(y,b;q_t)} \left\{u(y - b + q_t(b')[b' - (1-\lambda)b]) - u(y) + \beta \int \alpha \ln\left(1 + e^{S_{t+1}(y',b')/\alpha}\right) dF^y(y')\right\},$$

$$h_t(y, b) = \arg\max_{b' \in \Gamma(y,b;q_t)} \left\{u(y - b + q_t(b')[b' - (1-\lambda)b]) - u(y) + \beta \int \alpha \ln\left(1 + e^{S_{t+1}(y',b')/\alpha}\right) dF^y(y')\right\},$$

$$q_t(b') = \bar{p} \int \phi_{t+1}(y', b') \left[1 + (1-\lambda) q_{t+1}(h_{t+1}(y', b'))\right] dF^y(y'),$$

where $\phi_t(y', b') = e^{S_t(y',b')/\alpha}/(1 + e^{S_t(y',b')/\alpha})$*. Then the sequences* S_t*,* h_t*, and* q_t *converge uniformly to* S^∞*,* h^∞*, and* q^∞*, respectively. The limit tuple* $(S^\infty, h^\infty, q^\infty)$ *satisfies the associated fixed point system.*

Proof of Proposition 4. The value function sequence $\{S_t\}$ is pointwise monotone increasing from Proposition 3.3.1 and uniformly bounded from Lemma 1. Since each S_t is continuous from Lemma 2 and the domain is compact, Dini's theorem implies that the convergence $S_t \to S^\infty$ is uniform.

Since $S_t \to S^\infty$ uniformly, then the repayment probability $\phi_t \to \phi^\infty$ uniformly on $Y \times B$. Given this, consider the coupled system

$$\begin{aligned} h_t(y, b) &= \mathcal{H}(q_t)(y, b) \equiv \underset{b' \in \Gamma(y,b;q_t)}{\arg\max}\ \psi_t(y, b, b'), \\ q_{t-1}(b') &= \mathcal{Q}(h_t, q_t)(b') \equiv \bar{p} \int \phi_t(b', y') \Big[1 + (1-\lambda) q_t(h_t(b', y'))\Big]\, dF^y(y'). \end{aligned}$$

where $\Gamma_t(y, b; q_t) = \{b' \in B : 0 \leq y - b + q_t(b')[b' - (1-\lambda)b] \leq \bar{c}\}$ is continuous and compact under Proposition 2. The objective function for $t \leq T$ is given by

$$\psi_t(y, b, b') = u(y - b + q_t(b')[b' - (1-\lambda)b]) - u(y) + \beta \int \alpha \ln\left(1 + e^{S_{t+1}(y',b')/\alpha}\right) dF^y(y').$$

To establish uniform convergence of $\{q_t\}$, we need to ensure that the limit of any convergent subsequence also satisfies the fixed-point pricing equation. Since the policy operator $\mathcal{H}$ depends on q_t, we must control how the optimal bond choice changes as the prices vary. The next two lemmas formalize this continuity. Lemma 3 shows that the bond policy operator is continuous in the price function, and Lemma 4 shows that the pricing operator is continuous in both the bond policy and the price. These results allow us to pass to the limit in the recursive pricing equation for any subsequence.

Lemma 3 (Continuity of $\mathcal{H}$)**.** *If* $q_n \to q$ *uniformly on* B*, then*

$$\mathcal{H}(q_n) \to \mathcal{H}(q) \quad \textit{uniformly on } Y \times B.$$

Proof of Lemma 3. Since $Y \times B$ is compact, $\Gamma(y, b, q)$ is compact-valued and continuous in q, q is continuous from Lemma 2 and $\psi(y, b, b'; q)$ is continuous from Lemma 2 and strongly concave in b' from Proposition 2, Berge's maximum theorem implies that $\mathcal{H}(q)$ is continuous in (y, b, q). Uniform convergence follows from uniform convergence of q_n and compactness of the domain. $\square$

Lemma 4 (Continuity of $\mathcal{Q}$). *If $q_n \to q$ and $h_n \to h$ uniformly, then*

$$\mathcal{Q}(h_n, q_n) \to \mathcal{Q}(h, q) \quad \textit{uniformly on } B.$$

Proof of Lemma 4. For any $b' \in B$,

$$\begin{aligned}
|\mathcal{Q}(h_n, q_n)(b') - \mathcal{Q}(h, q)(b')| &\leq \bar{p}(1-\lambda) \int \phi^{\infty}(b', y') |q_n(h_n(b', y')) - q(h(b', y'))| \, dF^y(y') \\
&\leq \bar{p}(1-\lambda) \int \phi^{\infty}(b', y') \Big[|q_n(h_n(b', y')) - q(h_n(b', y'))| \\
&\qquad\qquad + |q(h_n(b', y')) - q(h(b', y'))| \Big] \, dF^y \\
&\leq \bar{p}(1-\lambda) \Big[\|q_n - q\|_{\infty} + \sup_{y, b'} |q(h_n(b', y)) - q(h(b', y))| \Big] \to 0,
\end{aligned}$$

using uniform convergence of q_n and h_n and uniform continuity of q on B. The bond price is continuous on B. Compactness of the domain upgrades continuity to uniform continuity. Uniformity therefore follows from uniform convergence of q_n and h_n together with uniform continuity of the bond price q. $\square$

Given uniform boundedness and equicontinuity in Lemmas 1 and 2, by the Arzelà–Ascoli theorem, the sequence $\{q_t\}$ admits a uniformly convergent subsequence, i.e. $q_{t_n} \to \bar{q}$ uniformly. By Lemma 3, $h_{t_n} = \mathcal{H}(q_{t_n}) \to \mathcal{H}(\bar{q}) \equiv \bar{h}$ uniformly. By Lemma 4, $q_{t_n-1} = \mathcal{Q}(h_{t_n}, q_{t_n}) \to \mathcal{Q}(\bar{h}, \bar{q})$ uniformly.

We now need to show that $\bar{q} = \mathcal{Q}(\mathcal{H}(\bar{q}), \bar{q})$ meaning that $\bar{q}$ is a stationary price function.[19] For this, observe that the price operator can be decomposed into two parts

$$\begin{aligned}
\|q_{t_n-1} - \bar{q}\|_{\infty} &= \|\mathcal{Q}(h_{t_n}, q_{t_n}) - \mathcal{Q}(\bar{h}, \bar{q})\|_{\infty} \\
&\leq \underbrace{\|\mathcal{Q}(h_{t_n}, q_{t_n}) - \mathcal{Q}(\bar{h}, q_{t_n})\|_{\infty}}_{A} + \underbrace{\|\mathcal{Q}(\bar{h}, q_{t_n}) - \mathcal{Q}(\bar{h}, \bar{q})\|_{\infty}}_{D}.
\end{aligned}$$

Regarding part D, for any $b' \in B$,

$$\begin{aligned}
|\mathcal{Q}(\bar{h}, q_{t_n})(b') - \mathcal{Q}(\bar{h}, \bar{q})(b')| &= \bar{p}(1-\lambda) \left| \int \phi^{\infty}(b', y') \left[q_{t_n}(\bar{h}(b', y')) - \bar{q}(\bar{h}(b', y')) \right] dF^y \right| \\
&\leq \bar{p}(1-\lambda) \int \phi^{\infty}(b', y') \, |q_{t_n}(\bar{h}(b', y')) - \bar{q}(\bar{h}(b', y'))| \, dF^y \\
&\leq \bar{p}(1-\lambda) \, \|q_{t_n} - \bar{q}\|_{\infty},
\end{aligned}$$

---

[19] This is because $q_{t_n-1} \neq q_{t_{n-1}}$. Hence q_{t_n-1} may not belong to the subsequence $\{q_{t_n}\}$.

where the last inequality uses $\bar{h}(b', y') \in B$ and $\phi^\infty \in [0, 1]$. Regarding part A, for any $b' \in B$,

$$
\begin{aligned}
|\mathcal{Q}(h_{t_n}, q_{t_n})(b') - \mathcal{Q}(\bar{h}, q_{t_n})(b')| &= \bar{p}(1-\lambda) \left| \int \phi^\infty(b', y') \left[q_{t_n}(h_{t_n}(b', y')) - q_{t_n}(\bar{h}(b', y')) \right] dF^y \right| \\
&\leq \bar{p}(1-\lambda) \int \phi^\infty(b', y') \left| q_{t_n}(h_{t_n}(b', y')) - q_{t_n}(\bar{h}(b', y')) \right| dF^y.
\end{aligned}
$$

Since q_{t_n} has a uniformly bounded first derivative $\|q_{b,t_n}\|_\infty \leq \bar{q}'$ from Lemma 1, it is Lipschitz with constant $\bar{q}'$. Moreover, from the implicit function theorem $h_{t_n,q}(y,b) \equiv \frac{\partial}{\partial q} h_{t_n}(y,b) = -\frac{\frac{\partial^2 \psi_{t_n}(y,b,h_{t_n}(y,b),q)}{\partial q \partial b'}}{\frac{\partial^2 \psi_{t_n}(y,b,h_{t_n}(y,b),q)}{\partial b'^2}}$. We have that $\frac{\partial^2 \psi_{t_n}(y,b,h_{t_n}(y,b),q)}{\partial q \partial b'} = u_{cc}(c_{t_n}(y,b))[q_{t_n}(b') + q_{t,b}(b')(b' - (1-\lambda)b)][b' - (1-\lambda)b] + u_c(c_{t_n}(y,b))$, which is uniformly bounded following Lemma 1. This together with the boundedness away from zero of $\partial^2_{b'b'}\psi_{t_n}$ due to the strong concavity from Proposition 2 ensures that h_{t_n} is Lipschitz in q_{t_n} with constant $\bar{h}'$. Hence

$$
|q_{t_n}(h_{t_n}(b', y')) - q_{t_n}(\bar{h}(b', y'))| \leq \bar{q}' \bar{h}' \|q_{t_n} - \bar{q}\|_\infty.
$$

Adding parts A and D together, we get

$$
\|q_{t_n - 1} - \bar{q}\|_\infty \leq \bar{p}(1-\lambda)(1 + \bar{q}'\bar{h}') \, \|q_{t_n} - \bar{q}\|_\infty.
$$

Given that $\{h_{t_n}\}$ is uniformly bounded, there is always an α large enough such that $\bar{p}(1-\lambda)(1+\bar{q}'\bar{h}') < 1$. Observe that $\phi_b(y,b) = \frac{1}{\alpha}\, \phi(1-\phi)\, \partial_b V^R$. We have $\phi(1-\phi) \leq \frac{1}{4}$ and $C_q \equiv \frac{1}{4}\|\partial_b V^R\|_\infty < \infty$ from Lemma 1 which gives the uniform bound $|\phi_b(y,b)| \leq \frac{C_q}{\alpha}$. Applying Leibniz rule to (4) and taking absolute values

$$
|q_b(b')| \leq \underbrace{\overline{p}\,|\chi_b(b')|}_{\leq \overline{p} C_q/\alpha} + \overline{p}(1-\lambda) \left[\underbrace{\int |\phi_b|\,|q(h)|\, dF^y}_{\leq C_q/(\alpha(r+\lambda))} + \underbrace{\int \phi\, |q_b(h)|\, |h_b|\, dF^y}_{\leq \|h_b\|_\infty \|q_b\|_\infty} \right],
$$

where we used the uniform bound on ϕ_b and $q \leq 1/(r+\lambda)$ for the first two terms, and $\phi \leq 1$ for the third. Taking the supremum over b' and collecting $\|q_b\|_\infty$

$$
\|q_b\|_\infty \Big[1 - \overline{p}(1-\lambda)\|h_b\|_\infty \Big] \leq \frac{\overline{p}\, C_q}{\alpha} \cdot \frac{1+r}{r+\lambda}. \tag{17}
$$

Whenever $\overline{p}(1-\lambda)\|h_b\|_\infty < 1$, one can always choose a large enough α to make $\bar{q}'$ small enough. To show this is the case, we can again use the implicit function theorem to get $h_{t_n,b}(y,b) \equiv \frac{\partial}{\partial b} h_{t_n}(y,b) = -\frac{\frac{\partial^2 \psi_{t_n}(y,b,h_{t_n}(y,b))}{\partial b \partial b'}}{\frac{\partial^2 \psi_{t_n}(y,b,h_{t_n}(y,b))}{\partial b'^2}}$. By Proposition 2 and Lemma 1, $h_{t_n,b}(y,b)$ is bounded. From Proposition 2, as $\alpha \to \infty$, $h_{t_n}(y,b) \to \max\{b' : \underline{y} - b' + \frac{1}{1+2r+\lambda}(b' - (1-\lambda)b') > 0\}$ implying that $\|h_b\|_\infty \to 0$. Given this and Equation (17), one can always choose a large enough α such that $\rho \equiv \bar{p}(1-\lambda)(1+\bar{q}'\bar{h}') < 1$. Given this,

$$\|q_{t_n-1} - \bar{q}\|_\infty \leq \rho \|q_{t_n} - \bar{q}\|_\infty \to 0,$$

so $q_{t_n-1} \to \bar{q}$ which implies that $\bar{q} = \mathcal{Q}(\mathcal{H}(\bar{q}), \bar{q})$ is indeed a stationary fixed point. As we considered a generic subsequence $\{q_{t_n}\}$, this is true for any convergent subsequence.

Since multiple subsequences could converge to different limits, we have to show the full sequence converges to the same limit. Suppose (h_1, q_1) and (h_2, q_2) are two stationary fixed points,

$$q_i(b') = \bar{p} \int \phi^\infty(b', y') \Big[1 + (1-\lambda) q_i(h_i(b', y')) \Big] \, dF^y(y'), \quad i = 1, 2.$$

Then, for any $b' \in B$,

$$\begin{aligned} |q_1(b') - q_2(b')| &= \bar{p}(1-\lambda) \Big| \int \phi^\infty(b', y') \big[q_1(h_1(b', y')) - q_2(h_2(b', y')) \big] \, dF^y \Big| \\ &\leq \bar{p}(1-\lambda) \int \phi^\infty(b', y') \, |q_1(h_1(b', y')) - q_2(h_2(b', y'))| \, dF^y \\ &\leq \bar{p}(1-\lambda) \, \|q_1 - q_2\|_\infty, \end{aligned}$$

where the last inequality comes from the assumption that (h_1, q_1) and (h_2, q_2) are two stationary fixed points. Taking the supremum over b' gives $\|q_1 - q_2\| \leq \bar{p}(1-\lambda)\|q_1 - q_2\|$. Since $\bar{p}(1-\lambda) < 1$, it follows that $\|q_1 - q_2\| = 0$, hence $q_1 = q_2 \equiv q^\infty$. Given q^∞, the strong concavity of $\psi^\infty(y, b, \cdot)$ from Proposition 2 and the compactness of the feasible set $\Gamma(y, b, q^\infty)$, $h^\infty(y, b) = \arg\max_{b' \in \Gamma(y,b,q^\infty)} \psi^\infty(y, b, b')$ is unique for each (y, b). Therefore the stationary bond policy is uniquely determined.

Since the price operator has a unique stationary fixed point, any stationary price function is q^∞. Therefore every convergent subsequence converges to q^∞, and thus the full sequence converges uniformly $q_t \to q^\infty$ on B. Since the bond policy corresponding to q^∞ is unique $h_t = \mathcal{H}(q_t) \to \mathcal{H}(q^\infty) = h^\infty$ uniformly on $Y \times B$. □

Proposition 5 (Smoothness). *If Assumption 1 holds, i.e.* $h_t(y,b) > 0$ *for all* (y,b) *and* $t < T$, ϕ_t, h_t

and q_t are of class $\mathcal{C}^\infty$ for all $t \leq T$.

Proof of Proposition 5. The proof goes by backward induction.

- At time T:

  There is no continuation value as the world ends. As a result, $h_T(y,b) = 0$ for all $y \in Y$ and all $b \in B$. The bond policy function $h_T(y,b)$ is trivially $\mathcal{C}^\infty$. However, the sovereign does not necessarily default for all $b > 0$. This depends on the realization of (ϵ^R, ϵ^A). The repayment probability is given by

  $$\phi_T(y,b) = \frac{e^{u(y-b(1+\frac{1-\lambda}{1+2r+\lambda}))/\alpha}}{e^{u(y-b(1+\frac{1-\lambda}{1+2r+\lambda}))/\alpha} + e^{u(y)/\alpha}}.$$

  We can derive an expression for its first derivative with respect to b

  $$\begin{aligned}\phi_{b,T}(y,b) &= -\frac{u_c(c_T)(1+\frac{1-\lambda}{1+2r+\lambda})}{\alpha}\frac{e^{u(y-b(1+\frac{1-\lambda}{1+2r+\lambda}))/\alpha}}{e^{u(y-b(1+\frac{1-\lambda}{1+2r+\lambda}))/\alpha} + e^{u(y)/\alpha}}\left[1 - \frac{e^{u(y-b(1+\frac{1-\lambda}{1+2r+\lambda}))/\alpha}}{e^{u(y-b(1+\frac{1-\lambda}{1+2r+\lambda}))/\alpha} + e^{u(y)/\alpha}}\right] \\ &= -\frac{u_c(c_T)(1+\frac{1-\lambda}{1+2r+\lambda})}{\alpha}\phi_T(y,b)[1-\phi_T(y,b)].\end{aligned}$$

  The differentiability of $\phi_T(y,b)$ directly depends on the differentiability of u. Given that u is $\mathcal{C}^\infty$, $\phi_T(y,b)$ is of class $\mathcal{C}^\infty$. Note that $\phi_T(y,0) = 1/2$ and ϕ_T is smooth at $b = 0$ since $c_T(y,b)$ is smooth in b. Given this, the bond price at time $T-1$ is

  $$q_{T-1}(b) = \frac{2}{1+2r+\lambda}\chi_T(b), \quad b \geq 0.$$

  Since $\chi_T(0) = 1/2$, we have $q_{T-1}(0) = \frac{1}{1+2r+\lambda}$, which equals the terminal buyback price in Assumption 2. Thus $q_{T-1}(b)$ is continuous at $b = 0$ and of class $\mathcal{C}^\infty$ for all $b \geq 0$ given the smoothness of $\phi_T(y,b)$.

- At time $T-1$:

  The optimal borrowing policy is determined by the GEE. In points such that $b_T > 0$, the optimal borrowing is the solution to (GEE),

  $$\begin{aligned}&g^{GEE}_{T-1}(b_T|b_{T-1},y) = \\ &u_c(c_{T-1})\left[q_{T-1}(b_T) + \int_{y'}\phi_{b,T}(y',b_T)\frac{2}{1+2r+\lambda}dF^y(b_T-(1-\lambda)b_{T-1})\right]\end{aligned}$$

$$- \beta \int_{y'} \phi_T(y', b_T) u_c(c_T) \left[1 + (1-\lambda) \frac{1}{1+2r+\lambda} \right] dF^y = 0.$$

The existence of the derivative $q_{b,T}$ and $h_{b,T}$ is ensured by the fact that q_T and h_T are both of class $\mathcal{C}^\infty$. The repayment probability ϕ_T is also $\mathcal{C}^\infty$. Moreover, the derivative of g_{T-1}^{GEE} with respect to b_T is invertible.[20] As a result, $g_{T-1}^{GEE}(b_T|b_{T-1}, y)$ is of class $\mathcal{C}^\infty$ and so is h_{T-1}.

The repayment probability is given by

$$\phi_{T-1}(y, b) = \frac{e^{V_{T-1}^R(y,b)/\alpha}}{e^{V_{T-1}^R(y,b)/\alpha} + e^{V_{T-1}^A(y)/\alpha}}.$$

As before, we can derive an expression for its first derivative with respect to b

$$\begin{aligned}
\phi_{b,T-1}(y, b) = & - \frac{u_c(c_{T-1})}{\alpha} [1 + (1-\lambda) q_{T-1}(h_{T-1})] \\
& \frac{e^{V_{T-1}^R(y,b)/\alpha}}{e^{V_{T-1}^R(y,b)/\alpha} + e^{V_{T-1}^A(y)/\alpha}} \left[1 - \frac{e^{V_{T-1}^R(y,b)/\alpha}}{e^{V_{T-1}^R(y,b)/\alpha} + e^{V_{T-1}^A(y)/\alpha}} \right] \\
= & - \frac{u_c(c_{T-1})}{\alpha} [1 + (1-\lambda) q_{T-1}(h_{T-1})] \phi_{T-1}(y, b) [1 - \phi_{T-1}(y, b)].
\end{aligned}$$

Given that u is $\mathcal{C}^\infty$, $\phi_{T-1}(y, b)$ is of class $\mathcal{C}^\infty$ for all $b \geq 0$. Given this repayment probability, the bond price at time $T-2$ reads

$$q_{T-2}(b) = \overline{p} \chi_{T-1}(b) + \overline{p}(1-\lambda) \int_{y'} \phi_{T-1}(y', b) q_{T-1}(h_{T-1}(y', b)) dF^y, \quad b \geq 0.$$

Since q_{T-1} is smooth, $q_{T-2}(b)$ is of class $\mathcal{C}^\infty$ for all $b \geq 0$.

- At time $T-n$ for $T \geq n \geq 2$:

  We develop an inductive argument for the remaining periods. We want to show that properties of the price function, the repayment probability and the borrowing policy are preserved from one iteration to the other for $T-n$ for all $T \geq n \geq 2$. We have previously shown that properties are preserved throughout T to $T-1$ (i.e. induction hypothesis).

  We have that q_{T-n} and ϕ_{T-n+1} are of class $\mathcal{C}^\infty$. Similarly, q_{T-n+1} and h_{T-n+1} are of class $\mathcal{C}^\infty$.

  As for every iteration, the optimal borrowing policy is determined by the GEE. In points such that

---

[20] We do not derive the expression for the derivative. The result is immediate after taking the derivative and the fact that the utility function is bijective. The same argument applies to all g functions considered in this section.

$b_{T-n+1} > 0$, the optimal borrowing is the solution to (GEE),

$$g^{GEE}_{T-n}(b_{T-n+1}|b_{T-n}, y) = u_c(c_{T-n})\Bigg[q_{T-n}(b_{T-n+1}) + (b_{T-n+1} - (1-\lambda)b_{T-n})$$

$$\Bigg\{\overline{p}(1-\lambda)\int_{y'}\phi_{T-n+1}(y', b_{T-n+1})q_{b,T-n+1}(h_{T-n+1}(y', b_{T-n+1}))h_{b,T-n+1}(y', b_{T-n+1})dF^y$$

$$+\,\overline{p}\int_{y'}\phi_{b,T-n+1}(y', b_{T-n+1})[1 + (1-\lambda)q_{T-n+1}(h_{T-n+1}(y', b_{T-n+1}))]dF^y\Bigg\}\Bigg] - \beta$$

$$\int_{y'}\phi_{T-n+1}(y', b_{T-n+1})u_c(c_{T-n+1})[1 + (1-\lambda)q_{T-n+1}(h_{T-n+1}(y', b_{T-n+1}))]dF^y = 0.$$

The existence of the derivative $q_{b,T-n+1}$ and $h_{b,T-n+1}$ is ensured by the fact that q_{T-n+1} and h_{T-n+1} are both of class $\mathcal{C}^\infty$. Moreover, the derivative of g^{GEE}_{T-n} with respect to b_{T-n} is invertible. As a result, $g^{GEE}_{T-n}(b_{T-n+1}|b_{T-n}, y)$ is of class $\mathcal{C}^\infty$ and so does h_{T-n} by application of the IFT. Thus, $h_{T-n}(y, b)$ is of class $\mathcal{C}^\infty$.

The repayment probability is given by

$$\phi_{T-n}(y, b) = \frac{e^{V^R_{T-n}(y,b)/\alpha}}{e^{V^R_{T-n}(y,b)/\alpha} + e^{V^A_{T-n}(y)/\alpha}}.$$

As before, given that u is $\mathcal{C}^\infty$ and $h_{T-n+1} > 0$, $\phi_{T-n}(y, b)$ is of class $\mathcal{C}^\infty$ for all $b \geq 0$. Given this repayment probability, the bond price at time $T-n-1$ reads

$$q_{T-n-1}(b) = \overline{p}\chi_{T-n}(b) + \overline{p}(1-\lambda)\int_{y'}\phi_{T-n}(y', b)q_{T-n}(h_{T-n}(y', b))dF^y, \quad b \geq 0.$$

Since q_{T-n} is smooth and $h_{T-n} > 0$ by assumption, $q_{T-n-1}(b)$ is of class $\mathcal{C}^\infty$ for all $b \geq 0$.

We therefore conclude that the properties of the price function, the repayment probability and the borrowing policy are preserved from one iteration to the other for $T-n$ with any $T \geq n \geq 2$.

□

Proposition 6 (Limiting Derivatives). *Let $\{h_{b,t}\}$ and $\{q_{b,t}\}$ denote the sequence of derivatives of $\{h_t\}$ and $\{q_t\}$, respectively. Under Assumption 1, $h_t \to h^\infty$ uniformly on $Y \times B$ and $q_t \to q^\infty$ uniformly on $Y \times B$ and large enough α, it holds that $h_{b,t} \to \frac{d}{db}h^\infty$ uniformly on $Y \times B$ and $q_{b,t} \to \frac{d}{db}q^\infty$ uniformly on $Y \times B$.*

Proof of Proposition 6. We prove the statement in two parts. First, we show the uniform convergence of the derivative of the bond policy. From the implicit function theorem, we have for $t < T$

$$h_{t,b}(y,b) \equiv \frac{\partial}{\partial b} h_t(y,b) = -\frac{\frac{\partial^2 \psi_t(y,b,h_t(y,b))}{\partial b \partial b'}}{\frac{\partial^2 \psi_t(y,b,h_t(y,b))}{\partial b'^2}},$$

where the objective function is given by

$$\psi_t(y,b,b') = u(y - b + q_t(b')[b' - (1-\lambda)b]) - u(y) + \beta \int \alpha \ln \left(1 + e^{S_{t+1}(y',b')/\alpha}\right) dF^y(y').$$

From Lemma 1, $\|\nabla^2 \partial_{b'} \psi_t(y,b,b')\| \leq P$. Therefore, for each $w_t(y,b,b') \in \{\nabla \partial_{b'} \psi_t(y,b,b')\} = \{(\frac{\partial^2 \psi_t}{\partial y \partial b'}, \frac{\partial^2 \psi_t}{\partial b \partial b'}, \frac{\partial^2 \psi_t}{\partial b'^2})\}$, for any two points (y_1, b_1, b_1'), (y_2, b_2, b_2') in $Y \times B \times B$, the mean value theorem implies

$$\left| w_t(y_1,b_1,b_1') - w_t(y_2,b_2,b_2') \right| \leq P \times \left\| (y_1,b_1,b_1') - (y_2,b_2,b_2') \right\|.$$

Hence, the sequence $\{\nabla \partial_{b'} \psi_t(y,b,b')\}$ is uniformly Lipschitz, and thus equicontinuous. Given this and the uniform boundedness shown in Lemma 1, the Arzelà-Ascoli theorem implies $\nabla \partial_{b'} \psi_t(y,b,b')$ admits a uniformly convergent subsequence on $Y \times B \times B$.

Given the formulation of $h_{t,b}$ from the implicit function theorem, the strict positivity of h_t from Assumption 1, the boundedness away from zero of $\partial^2_{b'b'} \psi_t$ due to the strong concavity and the uniform convergence of a subsequence of $\nabla \partial_{b'} \psi_t$, $h_{t,b}$ admits a uniformly convergent subsequence.

Since multiple subsequences could converge to different limits, we have to show the full sequence converges to the same limit. Suppose that along a subsequence $h_{n,b} \to g^\infty$ for some limit g^∞. Since $h_t \to h^\infty$ uniformly, we can integrate

$$h_t(y,b_1) - h_t(y,b_0) = \int_{b_0}^{b_1} h_{n,b}(y,b) db.$$

Taking $t \to \infty$, uniform convergence lets us pass the limit inside the integral

$$h^\infty(y,b_1) - h^\infty(y,b_0) = \int_{b_0}^{b_1} g^\infty(y,b) db.$$

Since the uniform limit of a continuous function is continuous, g^∞ is continuous. This implies that h^∞ is differentiable in b and $h^\infty_b = g^\infty$. Since every subsequence yields the same limit h^∞_b the full sequence converges uniformly, $h_{t,b} \to h_b^{\infty} = \frac{\partial}{\partial b} h^{\infty}$ on $Y \times B$.

Second, we show the uniform convergence of the derivative of the bond price. Recall that $\sup_{t,b'} \left| \frac{\partial^2 q_t(b')}{\partial b'^2} \right| \leq \bar{q}''$ for $b' > 0$ (the interior region selected by Assumption 1) in Lemma 1. As a result, for all $t < T$ and all b'_1, b'_2 in B, the mean value theorem implies that

$$|q_{t,b}(b'_1) - q_{t,b}(b'_2)| \leq \bar{q}'' \times |b'_1 - b'_2|$$

Hence, the sequence $\{q_{t,b}\}$ is uniformly Lipschitz, and thus equicontinuous. Given this and the uniform boundedness shown in Lemma 1, the Arzelà-Ascoli theorem implies $q_{t,b}$ admits a uniformly convergent subsequence. As before, we need to show the full sequence converges to the same limit. Since $q_t \to q^{\infty}$ uniformly, we can repeat the previous argument. As a result, every convergent subsequence of $\{q_{t,b}\}$ converges to the same limit. Hence, it holds that $q_{t,b} \to q_b^{\infty} = \frac{\partial}{\partial b} q^{\infty}$ uniformly on B. □

B The GEE in Alternative Settings

We formulate the GEE for three alternative settings, treated in parallel in the subsections below: the canonical quantitative model with long-term debt and persistent income of Hatchondo and Martinez (2009) and Chatterjee and Eyigungor (2012) (Appendix B.1); the partial-default model of Arellano et al. (2023) (Appendix B.2); and a positive recovery rate upon default (Appendix B.3), which nests an exogenous-size partial default as a special case. In each case, we highlight the modifications required relative to the baseline GEE derived in the main text, providing guidance for practitioners seeking to apply our methods to richer environments.

B.1 The Canonical Quantitative Model with Long-Term Debt

We have assumed an i.i.d endowment shock with permanent autarky upon default without endowment loss. However, in the canonical quantitative model with long-term debt, the endowment shock is persistent, there is an endowment loss upon default and a fixed market re-entry probability. Denote by $F^y(y'|y)$ the continuous distribution function of y' given y, by $\varphi(y)$ the endowment loss and $\xi \in [0, 1]$ the re-access probability. Given this, the value under financial autarky is then

$$V^A(y) = u(y - \varphi(y)) + \beta \int_{\underline{y}}^{\overline{y}} \left[\xi V^R(y', 0) + (1 - \xi) V^A(y') \right] \, dF^y(y'|y).$$

The value under repayment reads

$$V^R(y,b) = \max_{b'} \Big\{ u\left(y - b + q(y,b')\ \left[b' - (1-\lambda)b\right]\right) + \beta W(y,b') \Big\},$$

where $W(y,b') = \int \alpha \ln(e^{V^R(y',b')/\alpha} + e^{V^A(y')/\alpha})\, dF^y(y'|y)$. The definition of the repayment probability $\phi(y,b)$ remains unchanged. Given $\chi(y,b') = \int_y \phi(y',b')\ dF^y(y'|y)$, the price of one unit of bond reads

$$q(y,b') = \overline{p}\ \chi(y,b') + \overline{p}(1-\lambda) \int_{\underline{y}}^{\overline{y}} \phi(y',b')\ q\big[y', h(y',b')\big]\ dF^y(y'|y).$$

The main difference with our simplified environment is that the bond price and the expected continuation value depend on y due to the persistency of the endowment shock. Following the steps elicited in Section 5, the GEE can be formulated as

$$\begin{aligned} u_c(c)\Bigg[q(y,b') + (b' - (1-\lambda)b)\ \Bigg\{\overline{p}(1-\lambda)\int_{\underline{y}}^{\overline{y}} \phi(y',b')\ \mathcal{B}(y',b'|h,\phi,q)\ h_b(y',b')\ dF^y(y'|y) \\ + \overline{p}\int_{\underline{y}}^{\overline{y}} \phi_b(y',b')\ [1 + (1-\lambda)\ q(y',h(y',b'))]\ dF^y(y'|y)\Bigg\}\Bigg] \\ = \beta \int_{\underline{y}}^{\overline{y}} \phi(y',b')\ u_c(c')\ [1 + (1-\lambda)\ q(y',h(y',b'))]\ dF^y(y'|y). \end{aligned}$$

with $c = \mathcal{C}(y,b,h(y,b)|q) = y - b + q(y,h(y,b))(h(y,b) - (1-\lambda)b)$ and $c' = \mathcal{C}[y', h(y,b), h(y',h(y,b))|q]$. We substituted the derivative of the bond price using

$$\begin{aligned} \mathcal{B}(y,b|h,\phi,q) = \frac{1}{u_c(c)\ [h(y,b) - (1-\lambda)b]} \\ \times \Bigg[\beta\int_{\underline{y}}^{\overline{y}} \phi(y',h(y,b))\ u_c(c')\ [1 + (1-\lambda)q(y',h(y',h(y,b)))]\ dF^y(y'|y) \\ - u_c(c)q(y,h(y,b))\Bigg]. \end{aligned}$$

The GEE is very similar to the one derived in the main text since most of the changes affect the value of autarky and not the value of repayment. From a computational standpoint, the main adaptation required when moving from i.i.d. to persistent income is that all equilibrium objects—q, W, ϕ and their derivatives—become functions of y in addition to b'. This increases the dimensionality of the interpolation but does not change the structure of the algorithm. The pseudocode in Section 6.2 applies directly with the replacement of unconditional expectations by conditional ones. The re-entry probability ξ and endowment loss $\varphi(y)$ modify only the autarky value and do not affect the GEE itself, since the GEE characterizes the repayment decision conditional on not defaulting. A positive recovery rate is treated separately in Appendix B.3 below.

B.2 Partial Default

In the partial-default model, denote by a the total debt due. In a state (y, a), the borrower chooses new borrowing $b' \in B$ and partial default $d \in [0, 1]$ to maximize its value

$$\begin{aligned} V(a, y, z) = \max_{b,d} & \left\{ u\left(y - (1-d)a + q(a', d, z)b\right) + \beta \int V(a', y', z') dF_z(z'|z) \right\} \\ \text{s.t.} \quad & a' = \delta a + (1 + r - \delta)\kappa d a + b, \\ & y' = z'\Psi(d, z'). \end{aligned}$$

Any defaulted coupons da result in future obligations with present value of κda. Hence, the total debt due next period a' includes the accumulation of defaulted coupons after annuitization $(1 + r - \delta)\kappa$, the coupon payments due from the long-term legacy debt δa, and the new borrowing b'. Given this, the price of one unit of bond reads

$$\begin{aligned} q(a', d, z) = & \frac{1}{1+r} \int [1 - d(a', y', z')] \, dF_z(z'|z) \\ & + \frac{1}{1+r} \int [\delta + (1 + r - \delta)\kappa d(a', y', z')] \times q\left(a'', d(a', y', z'), z'\right) dF_z(z'|z), \end{aligned}$$

with $a'' = [\delta + (1 + r - \delta)\kappa d(a', y', z')]a' + b(a', y', z')$ and $y' = z'\Psi_d(d, z')$. To derive the GEE, define first the debt burden as $\Lambda(d', q') = 1 - d' + [\delta + (1 + r - \delta)\kappa d']q'$. The first-order conditions for an interior solution for the above maximization problem with respect to b and d are

$$u_c(c) \left[q(a', d, z) + q_a(a', d, z)b\right] = \beta \int u_c(c')\Lambda(d', q') \, dF_z(z'|z),$$

$$\begin{aligned} & u_c(c) \left[a + \left(q_a(a', d, z)(1 + r - \delta)\kappa a + q_d(a', d, z)\right) b\right] \\ & \qquad = \beta \int \left[u_c(c')(1 + r - \delta)\kappa a \, \Lambda(d', q') - z'\Psi_d(d, z')\right] dF_z(z'|z). \end{aligned}$$

As before, the derivatives of the bond price depend on derivatives of bond prices in future. By rearranging optimality conditions under equilibrium policies, the future derivatives of the bond price can be obtained by evaluating the following at future states:

$$q_a(a', d, z) = \mathcal{M}(a, y, z|b, d, q) \equiv \frac{\beta \int u_c(c')\Lambda(d', q')dF_z(z'|z) - q\, u_c(c)}{b\, u_c(c)},$$

$$q_a(a', d, z)(1 + r - \delta)\kappa a + q_d(a', d, z) = \mathcal{N}(a, y, z|b, d, q) \equiv \frac{\beta \int u_c(c')\big[(1 + r - \delta)\kappa a\, \Lambda(d', q') - z'\Psi_d(d, z')\big]\, dF_z(z'|z) - a\, u_c(c)}{b\, u_c(c)}.$$

B.3 Positive Recovery upon Default

We extend the persistent-income environment of Appendix B.1 to a positive recovery rate $\eta \in [0, 1]$ on the face value of debt. Upon default, creditors receive η per unit of defaulted face value. Whether this recovery is a costless transfer or is financed by the defaulting sovereign is governed by a single parameter $\theta \in [0, 1]$, the share of the recovery borne by the defaulter: at $\theta = 0$ the recovery is a stylized transfer that does not draw on the sovereign's resources, while at $\theta = 1$ the sovereign actually pays $\eta\, b$ to its creditors, so that default is *partial*—creditors recover η and absorb a haircut $1 - \eta$—with the recovery size exogenous. The latter is the constant-recovery case of the endogenous partial-default model of Arellano et al. (2023) formulated in Appendix B.2.

The value under default carries the funded share of the recovery as a flow cost,

$$V^A(y, b) = u\big(y - \varphi(y) - \theta\eta\, b\big) + \beta \int_{\underline{y}}^{\overline{y}} \big[\xi\, V^R(y', 0) + (1 - \xi)\, V^A(y', 0)\big]\, dF^y(y'|y),$$

where the continuation enters at zero debt, so b affects V^A only through the one-time payment $\theta\eta\, b$; write $c^A(y, b) = y - \varphi(y) - \theta\eta\, b$ for autarky consumption. At $\theta = 0$ this collapses to the baseline $V^A(y)$. The repayment value $V^R(y, b)$, the inclusive value $W(y, b')$, and the repayment problem are unchanged in form, while the repayment probability

$$\phi(y, b) = \frac{e^{V^R(y,b)/\alpha}}{e^{V^R(y,b)/\alpha} + e^{V^A(y,b)/\alpha}}$$

inherits whatever b-dependence V^A carries.

Because creditors receive η per unit in default regardless of how the recovery is financed, the bond price and its derivative do not depend on θ:

$$q(y, b') \;= \overline{p}\,\chi(y, b') + \overline{p}\,(1-\lambda) \int_{\underline{y}}^{\overline{y}} \phi(y', b')\, q(y', h(y', b'))\, dF^{y}(y'|y) + \overline{p}\,\eta\,\big[1 - \chi(y, b')\big],$$

$$\begin{aligned} q_b(y, b') \;&= \overline{p} \int_{\underline{y}}^{\overline{y}} \phi_b(y', b')\,\big[\,1 - \eta + (1-\lambda)\, q(y', h(y', b'))\big]\, dF^{y}(y'|y) \\ &\quad + \overline{p}\,(1-\lambda) \int_{\underline{y}}^{\overline{y}} \phi(y', b')\, q_b(y', h(y', b'))\, h_b(y', b')\, dF^{y}(y'|y), \end{aligned}$$

with $\chi(y, b') = \int \phi(y', b')\, dF^{y}(y'|y)$: the coefficient on ϕ_b becomes $1 - \eta + (1-\lambda)q$ in place of $1 + (1-\lambda)q$, and q gains the additive term $\overline{p}\,\eta[1-\chi]$.

The funded share enters the repayment first-order condition through the effect of b' on next period's default value, $\partial_{b'} V^{A}(y', b') = -\theta\eta\, u_c(c^{A}(y', b'))$. Define the recovery wedge

$$\Xi(y, b') \;\equiv\; \beta\,\theta\,\eta \int_{\underline{y}}^{\overline{y}} \big(1 - \phi(y', b')\big)\, u_c\big(c^{A}(y', b')\big)\, dF^{y}(y'|y), \qquad c^{A}(y', b') = y' - \varphi(y') - \theta\eta\, b',$$

the expected marginal cost of carrying debt into the default state. The repayment first-order condition reads

$$\begin{aligned} u_c(c)\,\big[q(y, b') + (b' - (1-\lambda)b)\, q_b(y, b')\big] = \beta \int_{\underline{y}}^{\overline{y}} \phi(y', b')\, u_c(c')\,\big[1 + (1-\lambda)\, q(y', h(y', b'))\big]\, dF^{y}(y'|y) \\ +\; \Xi(y, b'). \end{aligned}$$

Substituting q_b and replacing $q_b(y', h(y', b'))$ by the self-consistent representation, which inherits the wedge,

$$\begin{aligned} \mathcal{B}(y, b|h, \phi, q) = \frac{1}{u_c(c)\,[h(y, b) - (1-\lambda)b]} \bigg[\beta \int_{\underline{y}}^{\overline{y}} \phi(y', h)\, u_c(c')\,\big[1{+}(1{-}\lambda)q(y', h(y', h))\big]\, dF^{y}(y'|y) \\ +\; \Xi(y, h) - u_c(c)\, q(y, h)\bigg], \qquad h \equiv h(y, b), \end{aligned}$$

the GEE with recovery reads

$$
\begin{aligned}
u_c(c)\Bigg[q(y,b') + (b'-(1-\lambda)b)\Bigg\{ \overline{p}\,(1-\lambda)\int_{\underline{y}}^{\overline{y}} \phi(y',b')\,\mathcal{B}(y',b'|h,\phi,q)\,h_b(y',b')\,dF^y(y'|y) \\
+\,\overline{p}\int_{\underline{y}}^{\overline{y}} \phi_b(y',b')\left[1-\eta+(1-\lambda)\,q(y',h(y',b'))\right]dF^y(y'|y)\Bigg\}\Bigg] \\
= \beta\int_{\underline{y}}^{\overline{y}} \phi(y',b')\,u_c(c')\left[1+(1-\lambda)\,q(y',h(y',b'))\right]dF^y(y'|y) + \Xi(y,b'),
\end{aligned}
$$

with $c = \mathcal{C}(y,b,h(y,b)|q)$ and $c' = \mathcal{C}[y',h(y,b),h(y',h(y,b))|q]$ as before.

Two polar cases are worth isolating. With $\theta = 0$ the wedge vanishes and V^A is independent of b: the recovery is a costless transfer and the GEE differs from the baseline only through the modified pricing coefficient $1-\eta$. With $\theta = 1$ the sovereign funds the recovery in full, the wedge Ξ is active, and default is partial of exogenous size $1-\eta$. In either case $\eta = 0$ returns the baseline persistent-income GEE exactly.

The computational cost is mild for any θ. The autarky problem gains the state b, but only through the flow $u(c^A(y,b))$ with a b-independent continuation, so $V^A(y,b) = u(y-\varphi(y)-\theta\eta\, b)+\Omega(y)$ is as cheap to solve as the baseline. The pricing and derivative recursions are unchanged, and the recovery wedge $\Xi(y,b')$ is a known function of next-period debt once ϕ and autarky consumption are in hand. The pseudocode of Section 6.2 therefore applies with the modified pricing coefficient and an additive Ξ term; at $\sigma = 2$ the EGM closed-form quadratic for consumption is preserved, since η, θ, and Ξ enter only coefficients known on the EGM update step.

C Comparison with Chebyshev VFI

This appendix compares the GEE methods to a VFI implementation that uses a global Chebyshev approximation. The method solves the same Bellman system as discrete-state VFI, but replaces the tensor-grid representation of the repayment value with a global Chebyshev approximation to the surplus

$$
S\left(y,b\right) = V^R\left(y,b\right) - V^A\left(y\right).
$$

This provides a smoother VFI benchmark than discrete-state maximization while still relying on direct maximization over a debt-choice grid.

Algorithm C.1: Value Function Iteration with Chebyshev Interpolation

Let $\{b_i\}_{i=1}^{N_b^{\text{vfi}}}$ and $\{y_j\}_{j=1}^{N_y^{\text{vfi}}}$ denote Chebyshev nodes for debt and income, and let $\{b'_k\}_{k=1}^{N_{b'}^{\text{vfi}}}$ denote a fine grid for debt choices, and let $\pi_{j\ell}$ denote the income transition probability from y_j to y_ℓ; consumption

$c\left(y_j, b_i, b_k'; q\right)$ is defined as in Algorithm 1.

Step 1. Initialize. Set $n = 0$. Choose initial guesses for $S^{(0)}$, $V^{A,(0)}$, $q^{(0)}$, and $h^{(0)}$. Construct $\phi^{(0)}$ from Equation (11), equivalently

$$\phi^{(0)}\left(y, b\right) = \frac{\exp\left(S^{(0)}\left(y, b\right)/\alpha\right)}{1 + \exp\left(S^{(0)}\left(y, b\right)/\alpha\right)}$$

and recover $V^{R,(0)}\left(y, b\right) = V^{A,(0)}\left(y\right) + S^{(0)}\left(y, b\right)$.

Step 2. Maximize over the choice grid. For each Chebyshev state node $\left(y_j, b_i\right)$, choose

$$h^{(n+1)}\left(y_j, b_i\right) \in \arg\max_{b_k'} \Big\{ u\Big(y_j - \left[\lambda + (1-\lambda) z\right] b_i + q^{(n)}\left(y_j, b_k'\right)\left[b_k' - (1-\lambda) b_i\right]\Big) + \beta W^{(n)}\left(y_j, b_k'\right) \Big\},$$

treating choices with non-positive consumption as infeasible. The maximized objective defines $V^{R,(n+1)}\left(y_j, b_i\right)$.

Step 3. Update the autarky value. Update $V^{A,(n+1)}$ from the autarky equation.

Step 4. Update the surplus and fit the Chebyshev approximation. Set

$$S^{(n+1)}\left(y_j, b_i\right) = V^{R,(n+1)}\left(y_j, b_i\right) - V^{A,(n+1)}\left(y_j\right),$$

and fit a complete Chebyshev polynomial to $S^{(n+1)}$ by least squares on the state nodes.

Step 5. Update repayment probabilities and continuation values. Compute $\phi^{(n+1)}$ from Equation (11) and the inclusive continuation values $W^{(n+1)}$ from Equation (9), evaluating the surplus off grid through its Chebyshev approximation.

Step 6. Update prices. Set

$$q^{(n+1)}\left(y_j, b_k'\right) = \bar{p} \sum_{\ell} \pi_{j\ell} \phi^{(n+1)}\left(y_\ell, b_k'\right)\left[\lambda + (1-\lambda) z + (1-\lambda) q^{(n)}\left(y_\ell, h^{(n+1)}\left(y_\ell, b_k'\right)\right)\right].$$

The future policy and price terms are evaluated using the current interpolation objects.

Step 7. Apply policy inertia and check convergence. If the candidate policy at a state improves the repayment objective by less than ϵ_{pol}, retain the previous policy at that state. Stop when the sup-norm distances $\left\|q^{(n+1)} - q^{(n)}\right\|_\infty$ and $\left\|V^{R,(n+1)} - V^{R,(n)}\right\|_\infty$ fall below the tolerance; otherwise set $n \leftarrow n+1$ and return to Step 2.

Relative to discrete-state VFI, the substantive change is the representation of the repayment value. Chebyshev VFI still maximizes directly over a debt-choice grid, but it stores the value object through a smooth global approximation to the surplus. The method is therefore a VFI benchmark rather than a GEE method: it does not use first-order conditions and still requires a fine debt-choice grid. In the implementation, income is represented at Chebyshev nodes with conditional expectations evaluated by Gaussian quadrature, and—as for discrete-state VFI—a policy-inertia rule suppresses spurious cycling across adjacent debt choices.

The table shows that Chebyshev VFI produces similar stationary moments to the GEE methods, but substantially larger equilibrium residuals. Relative to benchmark Chebyshev VFI, the GEE price residuals are smaller by roughly four orders of magnitude across the reported price metrics, and the value residuals are smaller by four to five orders of magnitude. EGM also dominates Chebyshev VFI in runtime: benchmark EGM converges in 11.04 seconds, compared with 25.57 seconds for benchmark Chebyshev VFI. PI is slower because it solves the GEE by root-finding, but it delivers the same large accuracy gains relative to Chebyshev VFI. Thus, the comparison confirms that the GEE gains are not specific to the discrete-state VFI benchmark.

Table 6 reports the den Haan and Marcet (1994) accuracy test for all four methods, including VFI with Chebyshev interpolation. Under the test, Chebyshev VFI's absolute deviations from the nominal 5 percent rejection frequency are 0.65 and 3.40 percentage points (lower and upper tail) for the scalar instrument and 1.60 and 0.65 for the state-dependent instruments, intermediate between the GEE methods and discrete-state VFI.

Figure 5 shows the convergence paths of all four methods at the benchmark grid—the $\log_{10}$ sup-norm distance between successive iterates of the price function q and the repayment value V. The GEE methods have the cleanest and most regular profiles; discrete-state VFI and Chebyshev VFI converge under the same smoothing but with a longer initial transient.

C.1 Additional Grid Robustness

This appendix reports the full grid-robustness table underlying the main text. The table includes all four methods and all grid configurations. It is intentionally more detailed than the main comparison tables: it reports moments, price and value residuals, stationary-distribution-weighted residuals, internal grid sizes, iterations, and runtimes.

Several patterns are robust across grid specifications. First, the economic moments are stable for the accepted solutions. Across the GEE methods, the debt-to-output ratio remains 0.70, the average spread remains approximately 0.08, and the volatility and correlation moments vary little across the benchmark, coarse, and fine grids. The VFI methods produce broadly similar moments, although the spread and correlation moments are somewhat more sensitive to the grid specification.

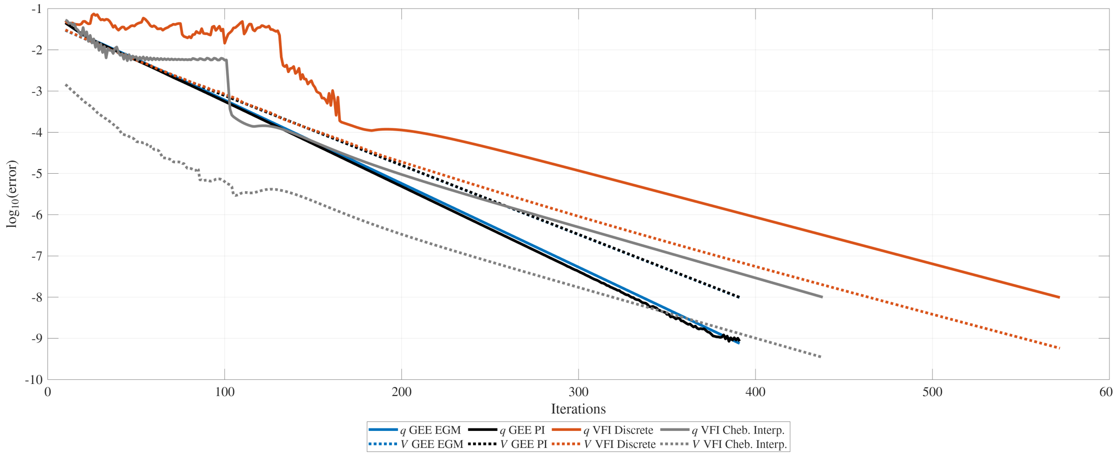

Figure 5: $\log_{10}$ sup-norm errors of the price function and the value of repayment after successive iterations of each algorithm at the benchmark grid.

Second, the GEE methods deliver large accuracy gains over the VFI Discrete Benchmark at every grid level. For EGM and PI, the price sup-norm residual is between roughly one and a half and three and a half orders of magnitude smaller than the VFI Discrete Benchmark residual across all grids; the value sup-norm residual is reduced by a factor of about 2.5 (Coarse) to 50 (Fine). Even Coarse GEE outperforms VFI Discrete Benchmark on every reported residual, indicating that the GEE methods are not relying on a fragile choice of internal grid.

Third, the VFI methods are more sensitive to grid design and computational cost. Discrete-state VFI requires much larger debt grids to obtain comparable accuracy, and runtime increases sharply as the debt grid is refined. Chebyshev VFI is faster than discrete-state VFI at the benchmark specification, but its value residuals remain much larger than those of the GEE methods. The fine Chebyshev VFI grid also substantially increases runtime. These robustness results reinforce the main message: the GEE methods achieve high residual accuracy with a smaller effective approximation space, while VFI-based methods require more intensive maximization or approximation to approach the same level of accuracy.

Table 5: Comparison with Chebyshev VFI

	VFI Cheb. Interp.	GEE PI		GEE EGM	
	Benchmark	Benchmark	Coarse	Benchmark	Coarse
Stationary moments — calibration targets					
b/y	0.70	0.70	0.70	0.70	0.70
Average spread	0.079	0.080	0.080	0.080	0.080
σ(spread)	0.048	0.045	0.047	0.045	0.047
corr(spread, y)	-0.79	-0.82	-0.79	-0.82	-0.79
Additional moments					
$\sigma(c)/\sigma(y)$	1.1	1.1	1.1	1.1	1.1
$\sigma(\frac{y-c}{y})/\sigma(y)$	0.14	0.13	0.14	0.13	0.14
corr(c, y)	0.99	0.99	0.99	0.99	0.99
corr$(\frac{y-c}{y}, y)$	-0.57	-0.60	-0.54	-0.60	-0.54
Numerical accuracy: VFI Cheb. column reports $\log_{10}$ *absolute residual;* *GEE columns report the ratio of the GEE residual to the corresponding Cheb VFI residual*					
Price sup	-1.56	0.00013	0.0015	0.00020	0.0012
Price L^2	-2.70	0.00015	0.0015	0.00022	0.0015
Price stat. L^2	-3.24	0.00042	0.0013	0.00029	0.0013
Value sup	-2.54	0.00016	0.0010	0.00018	0.00089
Value L^2	-2.88	0.000052	0.00033	0.000056	0.00034
Value stat. L^2	-3.19	0.000057	0.00024	0.000057	0.00023
Computational cost					
Grid y	35	101	101	101	101
Grid b	35	35	20	35	20
Grid b'	500	-	-	35	20
Polynomial order	35	-	-	-	-
Tolerance	1e-09	1e-09	1e-09	1e-09	1e-09
Iterations to converge	509	452	455	452	452
Time to converge	25.57	102.26	96.99	11.04	10.27
Time per iteration	0.050	0.226	0.213	0.024	0.023

Note: Times are in seconds. Moments are computed from the stationary distribution implied by each numerical solution. The benchmark Chebyshev VFI column reports accuracy as $\log_{10}$ unit-free residuals on the common validation grid. The GEE accuracy entries are normalized by the corresponding benchmark Chebyshev VFI residual: $e_m/e_{\text{Cheb}} = 10^{\log_{10}(e_m)-\log_{10}(e_{\text{Cheb}})}$. Thus, values below one indicate smaller residuals than benchmark Chebyshev VFI. Price residuals are normalized by the risk-free long-term bond price, and value residuals are relative to the policy-implied repayment value. For EGM, Grid b' is the exogenous next-period debt grid. For Chebyshev VFI, Grid b' is the fine choice grid used in the Bellman maximization. Comparing the same EGM residuals to the discrete-state VFI benchmark of Table 3 gives the ratios reported there; the absolute residuals are unchanged.

Table 6: den Haan and Marcet (1994) Accuracy Test — All Methods

$\omega(x_t) = 1$				$\omega(x_t) = [1, y_t, b_t]$			
VFI		GEE		VFI		GEE	
Discrete	Cheb. Interp.	PI	EGM	Discrete	Cheb. Interp.	PI	EGM
Absolute Deviation from Lower 5% (percentage points)							
1.10	0.65	0.05	0.00	1.85	1.60	0.45	0.20
Absolute Deviation from Upper 5% (percentage points)							
5.75	3.40	0.15	0.05	6.00	0.65	0.30	0.50

Table 7: Speed, Accuracy, and Moments across Grid Specifications

	VFI Discrete			VFI Cheb. Interp.			GEE PI			GEE EGM		
	Bench	Coarse	Fine	Bench	Coarse	Fine	Bench	Coarse	Fine	Bench	Coarse	Fine
Stationary moments — calibration targets												
b/y	0.70	0.70	0.70	0.70	0.69	0.70	0.70	0.70	0.70	0.70	0.70	0.70
Average spread	0.079	0.075	0.076	0.079	0.079	0.084	0.080	0.080	0.080	0.080	0.080	0.080
σ(spread)	0.042	0.044	0.039	0.048	0.052	0.049	0.045	0.047	0.044	0.045	0.047	0.045
corr(spread, y)	-0.83	-0.85	-0.83	-0.79	-0.74	-0.81	-0.82	-0.79	-0.83	-0.82	-0.79	-0.83
Additional moments												
$\sigma(c)/\sigma(y)$	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1	1.1
$\sigma(\frac{y-c}{y})/\sigma(y)$	0.13	0.13	0.13	0.14	0.14	0.13	0.13	0.14	0.13	0.13	0.14	0.13
corr(c, y)	1.0	1.0	0.99	0.99	0.99	1.0	0.99	0.99	1.0	0.99	0.99	1.0
corr$(\frac{y-c}{y}, y)$	-0.62	-0.63	-0.59	-0.57	-0.50	-0.60	-0.60	-0.54	-0.62	-0.60	-0.54	-0.62
Numerical accuracy: VFI Discrete Benchmark cell reports $\log_{10}$ *absolute residual;*												
all other cells report the ratio of the method's residual to the VFI Discrete Benchmark residual												
Price sup	-2.75	2.6	0.89	15	17	13	0.0020	0.023	0.00048	0.0032	0.019	0.00091
Price L^2	-4.03	1.9	0.85	21	23	20	0.0033	0.033	0.0021	0.0046	0.033	0.0010
Value sup	-5.11	1.9	1.7	370	360	370	0.058	0.38	0.020	0.066	0.33	0.023
Value L^2	-6.32	1.4	1.2	2800	2700	2800	0.14	0.90	0.047	0.15	0.91	0.048
Price stat. L^2	-4.14	1.8	0.83	7.9	11	7.4	0.0033	0.010	0.0032	0.0023	0.010	0.00052
Value stat. L^2	-6.61	2.1	1.7	2600	2800	2600	0.15	0.64	0.062	0.15	0.61	0.063
Computational cost												
Grid y	101	101	101	35	20	50	101	101	101	101	101	101
Grid b	350	200	500	35	20	50	35	20	50	35	20	50
Grid b'	-	-	-	500	500	500	-	-	-	35	20	50
Polynomial order	-	-	-	35	20	50	-	-	-	-	-	-
Tolerance	1e-09	1e-09	1e-09	1e-09	1e-09	1e-09	1e-09	1e-09	1e-09	1e-09	1e-09	1e-09
Iterations to converge	549	510	702	509	501	587	452	455	452	452	452	452
Time to converge	132.60	57.62	291.28	25.57	5.09	180.98	102.26	96.99	108.76	11.04	10.27	11.50
Time per iteration	0.242	0.113	0.415	0.050	0.010	0.308	0.226	0.213	0.241	0.024	0.023	0.025

Note: Times are in seconds. The VFI Discrete Benchmark cell of each accuracy row reports the $\log_{10}$ unit-free residual on a common validation grid. All other accuracy cells report the ratio of the method's residual to the VFI Discrete Benchmark residual: $e_m/e_{\mathrm{VFI,B}} = 10^{\log_{10}(e_m)-\log_{10}(e_{\mathrm{VFI,B}})}$. Values below one indicate smaller residuals than VFI Discrete Benchmark. Price residuals are normalized by the risk-free long-term bond price, and value residuals are relative to the policy-implied repayment value. The stationary L^2 rows weight squared residuals by the stationary distribution of the income process under the Tauchen approximation, applied uniformly across the debt dimension. Grid b' denotes the exogenous next-period debt grid for EGM and the fine choice grid for Chebyshev VFI; discrete-state VFI uses the same grid for inherited and next-period debt. Polynomial order applies only to Chebyshev VFI.